Particle transport in a random velocity field with Lagrangian statistics

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The transport properties of a random velocity field with Kolmogorov spectrum and time correlations defined along Lagrangian trajectories are analyzed. The analysis is carried out in the limit of short correlation times, as a perturbation theory in the ratio, scale by scale, of the eddy decay and turnover time. Various quantities such as the Batchelor constant and the dimensionless constants entering the expression for particle relative and self-diffusion are given in terms of this ratio and of the Kolmogorov constant. Particular attention is paid to particles with finite inertia. The self-diffusion properties of a particle with Stokes time longer than the Kolmogorov time are determined, verifying on an analytical example the dimensional results of Olla [Phys. Fluids **14**, 4266 (2002)]. Expressions for the fluid velocity Lagrangian correlations and correlation times along a solid particle trajectory are provided in several parameter regimes, including the infinite Stokes time limit corresponding to Eulerian correlations. The concentration fluctuation spectrum and the nonergodic properties of a suspension of heavy particles in a turbulent flow, in the same regime, are analyzed. The concentration spectrum is predicted to obey, above the scale of eddies with lifetime equal to the Stokes time, a power law with universal $-4/3$ exponent, and to be otherwise independent of the nature of the turbulent flow. A preference of the solid particle to lie in less energetic regions of the flow is observed.

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I. INTRODUCTION

One of the differences between high Reynolds number turbulence and other examples of random fields with power-law scaling, is the Lagrangian nature of time correlations [1]. From the theoretical point of view, the need for a Lagrangian treatment of time correlations has been one of the main difficulties in the realization of statistical turbulent closures [2]. Because of this, many such theories assume from the start that the turbulence dynamics be equivalent to that of a random velocity field with identical energy spectrum but Eulerian time statistics, i.e., the fluctuations decay without being transported by the larger vortices [3–5]. Such an assumption does not work in the case of particle transport: both relative and self-diffusion are affected by the way in which time correlations are defined.

Concerning self-diffusion, in Kolmogorov turbulence, fluctuations at a scale l within the inertial range, have characteristic velocity $\sim l^{1/3}$ and decay time $\sim l^{-2/3}$ along fluid trajectories. Hence, in a time t the velocity of a fluid parcel will change by an amount of the order of that of a fluctuation with that lifetime, i.e., by $t^{1/2}$. If the fluctuations were not advected by the flow, the fluid parcel would see the fluctuation only for the time $\sim l^{-1}$ it takes to cross it. The variation of the fluid parcel velocity in a time t would be therefore $\sim t^{1/3}$.

Concerning relative diffusion, this process is determined by vortices with the size of the fluid parcel separation at the given time. If these vortices were fixed in space, their effect on relative diffusion would be proportional to the crossing time by the fluid parcels, which is determined by the large scale properties of the flow. In other words, if time correlations were given in an Eulerian reference frame, the process of relative diffusion would not depend solely on the interparticle distance and on the velocity difference, but also on the total velocity.

Given the difficulty in defining a velocity field with Lagrangian statistics, a successful strategy for the treatment of transport has been to neglect time correlations altogether, i.e., to consider a velocity field $\mathbf{u}$ such that $\langle u_\alpha(\mathbf{x},t)u_\beta(0,0)\rangle = U_{\alpha\beta}(\mathbf{x})\delta(t)$: the so called Kraichnan model [6]. In this model, Eulerian and Lagrangian time statistics trivially coincide in what is the zero order of some perturbation theory in powers of the correlation time of the turbulence. It has been possible, in particular, to determine the anomalous scaling exponents of a passive scalar injected at large scales in the velocity field [7–10]. The origin of this success is that, although the time structure of the velocity correlation is lost, that of the relative displacement, whose geometrical properties determine the passive scalar correlations, is preserved [11–13]. (For instance, particle pair separation still obeys Richardson diffusion.)

The question, at this point, is how to introduce finite correlation times in a perturbative manner, but preserving the Lagrangian nature of correlations. There are practical reasons to do this. One motivation, of course, is to be able to determine the time correlations of the particle velocities. Lagrangian dispersion models [14–16] are based on the adoption of prescriptions on the form of these time correlations; to be able to determine them directly from the statistical properties of the velocity field would be, therefore, of some interest.

It must be said that most of the prescriptions entering a Lagrangian dispersion model could be obtained, in practice, by dimensional reasoning or by experiments. In some cases, like in the presence of particles endowed with inertia, this turns out, however, to be a difficult task [17,18]. It is very difficult, for instance, to make assumptions on the preference of solid particles to lie in certain regions of the flow instead of others [19–21]. Solid particle transport by a turbulent flow is an example of a situation in which careful treatment of the time dependent statistics of the velocity field is essential. It is precisely the interplay between the response time of the solid particle to the fluid, i.e., the Stokes time τ_S, and the characteristic times of the turbulent flow [22], which determines the dynamics, and this is clearly lost when all the turbulent times are sent to zero.

Recently, there has been strong theoretical interest on the problem of turbulence induced concentration fluctuations in a heavy particle suspension. In Ref. [23], the role of a finite correlation time of the turbulent field was recognized. In Ref. [24], the case of a particle with Stokes time in the turbulent viscous range was analyzed exploiting the fact that, in this case, the fluid velocity is spatially smooth on the scale of interest for the solid particle. In both Refs. [23] and [24], however, the inertial range structure of the turbulent flow was disregarded altogether. The approach carried on here, allows one instead to analyze the production of concentration fluctuations in any regimes of Stokes times, in particular, in the inertial range, where qualitatively different behaviors for the concentration fluctuation buildup are observed.

The purpose of this paper is to extend the Kraichnan model to short but finite correlation times, preserving, in a controlled perturbation theory, the Lagrangian structure of correlations, and providing several applications to the transport of particles with and without inertia. The analysis will be confined to a situation of two-dimensional, stationary, homogeneous, and isotropic turbulence.

This paper is organized as follows. In Sec. II, the equations determining the extension of the Kraichnan model will be illustrated and their main properties discussed. Section III will be devoted to the dynamics of passive tracers; the self-diffusion and relative diffusion of fluid parcels, including the expression for the constants involved, will be determined; the effect of finite diffusivity will be discussed and the Batchelor constant for a passive scalar injected at large scale in the flow will be calculated. In Sec. IV, the transport properties of a heavy particle with Stokes time longer than the Kolmogorov time will be studied, focusing on the relation between the correlation time for the fluid velocity sampled by the particle, and its Lagrangian and Eulerian counterparts. Section V will be devoted to calculation of the concentration fluctuations arising from compressibility of the heavy particle flow. In Sec. VI, the bias introduced by inertia in the sampling of fluid velocity by solid particles (nonergodic effects) will be analyzed. Section VII will be devoted to conclusions.

II. FINITE CORRELATION TIME EXTENSION OF THE KRAICHNAN MODEL

A two-dimensional random velocity field with Eulerian correlation times scaling like the eddy turnover time of a real turbulent flow can be obtained very simply, writing appropriate Langevin equations for the Fourier components of the vorticity field,

$$\partial_t q_{\mathbf{k}}(t)+\gamma_k q_{\mathbf{k}}(t)=h_k \xi_{\mathbf{k}}(t), \tag{2.1}$$

where $q_{\mathbf{k}}(t)$ is the (space) Fourier transform of the vorticity,

$$q(\mathbf{x},t)=\boldsymbol{\nabla}_\perp\cdot\mathbf{u}(\mathbf{x},t), \tag{2.2}$$

[for the generic vector $\mathbf{v}$, we indicate $\mathbf{v}_\perp=(-v_2,v_1)$], and $\xi_{\mathbf{k}}(t)$ is the Fourier transform of a zero mean fully uncorrelated noise term of unitary amplitude,

$$\langle\xi(\mathbf{x},t)\xi(0,0)\rangle=\delta(\mathbf{x})\,\delta(t). \tag{2.3}$$

The damping and forcing kernels γ_k and h_k are chosen, for $k\ll\eta^{-1}$, with η the Kolmogorov length of the flow, as

$$\gamma_k=\rho C_{Kol}^{1/2}\bar{\epsilon}^{1/3}(k^2+k_0^2)^{1/3}, \tag{2.4}$$

$$H_k=|h_k|^2=\frac{8\pi\rho C_{Kol}^{3/2}\bar{\epsilon}k^2}{k^2+k_0^2}, \tag{2.5}$$

while, for $k\eta>1$, some cutoff is imposed on the forcing amplitude H_k. In this way, the velocity spectrum $\mathbf{U}_{\mathbf{k}}(t)$, defined by $\langle\mathbf{u}_{\mathbf{k}}(t)\mathbf{u}_{\mathbf{p}}(0)\rangle=\mathbf{U}_{\mathbf{k}}(t)(2\pi)^2\ \delta(\mathbf{k}+\mathbf{p})$, will read, for $k\eta\ll 1$,

$$\mathbf{U}_{\mathbf{k}}(t)=4\pi C_{Kol}\bar{\epsilon}^{2/3}\frac{\mathbf{k}_\perp\mathbf{k}_\perp}{k^2}\frac{\exp(-\gamma_k|t|)}{(k^2+k_0^2)^{4/3}}, \tag{2.6}$$

where C_{Kol} and $\bar{\epsilon}$ play the role, respectively, of the Kolmogorov constant and the inertial range energy flux in a real turbulent field having this correlation spectrum. For $k_0\ll k\ll\eta^{-1}$, we thus have the energy spectrum: $E_k=C_{Kol}\bar{\epsilon}^{2/3}k^{-5/3}$. Identifying γ_k^{-1} with the decay time and $k^{-2}U_k^{-1/2}(0)$ with the turnover time of an eddy at scale k^{-1}, we see that ρ gives the ratio of the eddy turnover and eddy decay time in the inertial range. The effect of sweep by the large scales, however, is not accounted for in this way.

The most natural way to impose Lagrangian correlations in the random velocity field is to include an advection term in Eq. (2.1), which will take the following form in real space:

$$[\partial_t+\mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]q(\mathbf{x},t)+\int d^2y\,\gamma(\mathbf{x}-\mathbf{y})q(\mathbf{y},t)$$
$$=\int d^2y\,h(\mathbf{x}-\mathbf{y})\xi(\mathbf{y},t). \tag{2.7}$$

This has the form of a vorticity equation in which the forcing and dissipation terms, instead of being localized, respectively, at large and small scales, act over the whole of the inertial range, and this is reflected in their being nonlocal operators in real space. This is opposite to what happens in a real turbulent field, where energy balance is established between large scale forcing and small scale viscous dissipation, by means of the nonlinear cascade. A nonlinear cascade is still present because of the convection term, but it acts on the time scale of the eddy turnover time, and, for large ρ, its effect is only a correction to that of the forcing and damping terms. Choosing ρ large has, therefore, the consequence that convection acts merely as a large scale sweep.

Actually, Eq. (2.7) looks a lot like the typical starting point of many turbulent closures [3–5], in which γ_k gives the turbulent response function (eddy viscosity of small scales) and h_k the nonlinear forcing by the cascade. For instance,

ρ^{-2} coincides with the renormalized dimensionless coupling constant of the renormalization group (RNG) closure [5,25], and its smallness is there the basis for the establishment of a perturbation theory. Here, the philosophy is rather different: no parametrization of the turbulence cascade is sought, ρ is chosen arbitrarily large, and the similarity with real turbulence is expected to be only kinematic. (Also, the separation of a Kolmogorov constant out of the energy flux $\overline{\epsilon}$ is arbitrary.)

Things can be made a little bit more quantitative, introducing scale by scale the sweep time,

$$T_k = k^{-1}\langle u^2\rangle^{-1/2} \sim C_{Kol}^{-1/2}\overline{\epsilon}^{-2/3} k_0^{1/3} k^{-1}, \tag{2.8}$$

i.e., the time needed to a vortex of size k^{-1} to pass in front of a fixed probe. We see that sweep is important for all scales for which $\gamma_k T_k < 1$, i.e., from Eqs. (2.4) and (2.6), for $k > k_0\rho^3$. The Kraichnan model is recovered when sweep can be neglected in all of the inertial range, i.e., for $\rho > (\eta k_0)^{-1/3}$. This means basically that the zero correlation time limit is taken before the infinite Reynolds number limit $\eta k_0 \to 0$. In this regime we have

$$\mathbf{U}_{\mathbf{k}}(t) \simeq \frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k^2}\frac{2\pi}{\rho} C_{Kol}^{1/2}\overline{\epsilon}^{1/3} k^{-10/3}\delta(t). \tag{2.9}$$

To understand what happens in the regime of dominant sweep, it is convenient to shift to Lagrangian coordinates. Introduce then the coordinate $\mathbf{z}(t|\mathbf{x},t_0)$ of a fluid parcel which at time t_0 is at $\mathbf{x}$, and define the Lagrangian velocity,

$$\mathbf{u}^L(\mathbf{x},t) = \mathbf{u}(\mathbf{z}(t|\mathbf{x},0),t) \tag{2.10}$$

and analogous expressions for $q^L(\mathbf{x},t)$ and the other fields. After introducing the increase of trajectory separation in a time t: $\delta\mathbf{z}(t|\mathbf{x},\mathbf{y}) = \mathbf{z}(t|\mathbf{x},0) - \mathbf{z}(t|\mathbf{y},0) - (\mathbf{x}-\mathbf{y})$, Eq. (2.7) becomes, in the new variables,

$$\partial_t q^L(\mathbf{x},t) + \int d^2y\, \gamma[\mathbf{x}-\mathbf{y}+\delta\mathbf{z}(t|\mathbf{x},\mathbf{y})] q^L(\mathbf{y},t) = \int d^2y\, h[\mathbf{x}-\mathbf{y}+\delta\mathbf{z}(t|\mathbf{x},\mathbf{y})]\xi(\mathbf{y},t), \tag{2.11}$$

which must be coupled with the equation for $\delta\mathbf{z}$; inverting Eq. (2.2),

$$\partial_t \delta\mathbf{z}(t|\mathbf{x},\mathbf{y}) = \frac{1}{2\pi}\int d^2r [\mathbf{G}(\mathbf{x},\mathbf{r}) - \mathbf{G}(\mathbf{y},\mathbf{r})] q^L(\mathbf{r},t) \tag{2.12}$$

with

$$\mathbf{G}(\mathbf{x},\mathbf{r}) = \frac{[\mathbf{x}-\mathbf{r}+\delta\mathbf{z}(t|\mathbf{x},\mathbf{r})]_\perp}{|\mathbf{x}-\mathbf{r}+\delta\mathbf{z}(t|\mathbf{x},\mathbf{r})|^2}. \tag{2.13}$$

We see then that the natural expansion parameter of the theory is

$$\frac{\delta z(t|\mathbf{x},\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \sim \frac{|\mathbf{u}(\mathbf{x},0) - \mathbf{u}(\mathbf{y},0)|}{|\mathbf{x}-\mathbf{y}|\,\gamma_{|\mathbf{x}-\mathbf{y}|^{-1}}} \sim \rho^{-1}, \tag{2.14}$$

i.e., the relative amount of particle separation increase in an eddy lifetime. The zero order of the theory, which is Gaussian and is described by Eq. (2.6) after substituting $\mathbf{u} \to \mathbf{u}^L$, corresponds to neglecting trajectory separation in an eddy lifetime, while keeping the uniform large scale sweep, implicit in the Lagrangian field q^L.

Although the results that follow in the present paper are all obtained to the lowest order in the ρ expansion [26], associated with neglecting all non-Gaussian effects in $\mathbf{u}$, a diagrammatic expansion of Eqs. (2.12) and (2.13) in terms of the fields q^L, $\delta\mathbf{z}$ and their conjugate could be obtained by means of the Martin-Siggia-Rose formalism [27]. This expansion would only be valid locally around $t=0$, since, at long times, trajectory separation becomes dominant. (To be consistent, this perturbation expansion should not receive contribution by correlations involving pairs of points in space-time such that $\gamma_{|\mathbf{x}-\mathbf{x}'|^{-1}}|t-t'| > \rho$, but this is expected to be true from the exponential decay of the time correlations.)

The interaction terms in the perturbation expansion are obtained Taylor expanding the kernels γ, $\mathbf{G}$, and h (H working with the field action). The result for γ is, for instance,

$$\gamma(\mathbf{x}-\mathbf{y}+\delta\mathbf{z}(t|\mathbf{x},\mathbf{y})) = \gamma(\mathbf{x}-\mathbf{y}) + \sum_{n=1}^{\infty} \lambda_{\gamma_n}\gamma_n^{i_1\cdots i_n}(\mathbf{x}-\mathbf{y}) \times \delta z_{i_1}(t|\mathbf{x},\mathbf{y})\cdots\delta z_{i_n}(t|\mathbf{x},\mathbf{y}), \tag{2.15}$$

with $\lambda_{\gamma_n}=1$ a coefficient that may scale when carrying on power counting. Similar coefficients λ_{G_n} and λ_{H_n} are introduced in the Taylor expansion for G and H. The theory is thus characterized by an infinite number of interactions involving vertices, which, to $O(\rho^{-n})$, have up to $2+n$ legs.

To check for divergences at large k in the perturbation expansion, we use power counting directly in Eqs. (2.11)–(2.13) [28]. Rescaling coordinates and times as

$$x \to \Lambda x \quad \text{and} \quad t \to \Lambda^{2/3} t, \tag{2.16}$$

Eqs. (2.11)–(2.13) remain invariant in form, provided that we rescale the various fields and interactions $A = q^L, \delta z, \lambda_{\gamma_n}, \lambda_{G_n}, \lambda_{H_n}$: as $A \to \Lambda^{[A]}A$, with

$$[q^L] = -\frac{2}{3}, \quad [\delta\mathbf{z}] = 1,$$

$$[\lambda_{\gamma_n}] = [\lambda_{H_n}] = [\lambda_{G_n}] = 0. \tag{2.17}$$

This leads to expect logarithmic divergences at large k, meaning renormalizability of the field theory and the possibility of logarithmic correction to scaling, produced by renormalization of the parameters in Eqs. (2.11)–(2.13).

It must be mentioned that marginal interactions and renormalizability are consequences of the dimensional relation implicit in Kolmogorov scaling: $[q^L]=-[t]$. In general, had we set

$$\gamma_k\sim k^r,\quad H_k\sim k^s, \tag{2.18}$$

we would have obtained

$$[q^L]=\frac{r-s-2}{2},\quad [\delta\mathbf{z}]=\frac{3r-s}{2},$$

$$[\lambda_{\gamma_n}]=[\lambda_{H_n}]=[\lambda_{G_n}]=-n([t]+[q^L])=\frac{n}{2}(s+2-3r). \tag{2.19}$$

We thus see that super-renormalizability $[\lambda]<0$ and nonrenormalizability $[\lambda]>0$ of the theory occur, respectively, for positive and negative $[q^L]+[t]$ [28]. This corresponds to the two regimes of eddy decay time becoming asymptotically longer (shorter) than the eddy turnover time, and hence the nonlinearity becoming dominant (negligible) at large scales.

Marginality of the interactions means that logarithmic divergences may arise both at large and small k. At small k, however, such divergences are not expected, due to the subtraction in the definition of $\delta\mathbf{z}$. The reason is sketched below (more details will be given in a separate publication; it must be said, anyway, that this is not a surprise: Lagrangian closures [2] were introduced precisely to cure the infrared divergences arising in the original Eulerian theories). As it appears from Eq. (2.17), small k divergence is due to internal lines in a loop diagram involving the field $\delta\mathbf{z}$. The scaling of Eq. (2.17) is associated with large, not with small k behaviors. In fact, the divergences occurring for large k in a loop diagram will not change if we exchange $\delta\mathbf{z}(t|\mathbf{x},\mathbf{y})\to\mathbf{z}(t|\mathbf{x},0)+\mathbf{z}(t|\mathbf{y},0)-(\mathbf{x}+\mathbf{y})$; this is because each small eddy contributes to the separation $\mathbf{x}-\mathbf{y}$ an amount that is of the same order of the one to sweep. Now, the logarithmic divergence predicted at small k in a loop diagram comes indeed, from equating the scaling of the sweep $\mathbf{z}(t|\mathbf{x},0)+\mathbf{z}(t|\mathbf{y},0)-(\mathbf{x}+\mathbf{y})$ with that of the trajectory separation $\delta\mathbf{z}(t|\mathbf{x},\mathbf{y})$, also at small k, which is incorrect. For small k, this scaling should be corrected by a factor k per field $\delta\mathbf{z}$ involved in the lines of the loop, and this is enough to eliminate divergence.

III. PASSIVE TRACER TRANSPORT

A. Self-diffusion of a fluid parcel

Lagrangian correlation functions in the form $\langle\mathbf{u}^L(\mathbf{x},t)\mathbf{u}^L(\mathbf{x},0)\rangle=\langle\mathbf{u}(\mathbf{z}(t|\mathbf{x},0),t)\mathbf{u}(\mathbf{x},0)\rangle$ are the simplest objects one may try to calculate from the random velocity field introduced in Sec. II. The starting point, to lowest order in ρ^{-1}, and after sending the Kolmogorov scale η to zero, is the following modification of Eq. (2.6):

$$\mathbf{U}^L_{\mathbf{k}}(t)=4\pi C_{Kol}\overline{\epsilon}^{2/3}\frac{\mathbf{k}_\perp\mathbf{k}_\perp}{k^2}\frac{\exp(-\gamma_k|t|)}{(k^2+k_0^2)^{4/3}}. \tag{3.1}$$

The Lagrangian correlation time τ_L is then readily calculated,

$$\tau_L^{-1}=\langle|\mathbf{u}^L|^2\rangle\left[\int dt\langle\mathbf{u}^L(\mathbf{x},t)\cdot\mathbf{u}^L(\mathbf{x},0)\rangle\right]^{-1}=2\rho C_{Kol}^{1/2}\overline{\epsilon}^{1/3}k_0^{2/3} \tag{3.2}$$

and we have the following relation between the turbulence level $u_T^2=\langle u^2\rangle=\langle|\mathbf{u}^L|^2\rangle$ and the integral scales of the flow k_0 and τ_L:

$$u_T^2=3C_{Kol}\overline{\epsilon}^{2/3}k_0^{-2/3}=6\rho C_{Kol}^{3/2}\overline{\epsilon}\tau_L. \tag{3.3}$$

The correlation time τ_L is determined by the particular form of $\mathbf{U}^L_{\mathbf{k}}$ we have chosen at small k, which is nonuniversal. It is more interesting, and relevant from the point of view of Lagrangian dispersion modeling [14,29], to calculate the Lagrangian time structure function,

$$\langle[u_\alpha^L(\mathbf{x},t)-u_\alpha^L(\mathbf{x},0)][u_\beta^L(\mathbf{x},t)-u_\beta^L(\mathbf{x},0)]\rangle$$
$$=\frac{1}{2}\langle|\mathbf{u}^L(\mathbf{x},t)-\mathbf{u}^L(\mathbf{x},0)|^2\rangle\delta_{\alpha\beta}. \tag{3.4}$$

We discover immediately that, in order to have a self-similar spectrum for the inertial range, the time correlations should have continuous time derivative at $t=0$, a property not satisfied by Eq. (3.1).

This self-similarity violation can be illustrated in a simple way, imagining the turbulence field in the neighborhood of the fluid parcel as a superposition of nested eddies with scale l_n, velocity u_n, and eddy turnover time τ_n,

$$l_n=l_0 2^{-n},\quad u_n=u_0 2^{-n/3},\quad \tau_n=\tau_0 2^{-2n/3}. \tag{3.5}$$

If the time correlation decayed linearly for $t\to0$, we would have

$$\langle|\mathbf{u}^L(\mathbf{x},t)-\mathbf{u}^L(\mathbf{x},0)|^2\rangle\sim\sum_{\tau_n<t}u_n^2\frac{t}{\tau_n}+\sum_{\tau_n>t}u_n^2\sim u_0^2\ln(\tau_0/t)\frac{t}{\tau_0}. \tag{3.6}$$

Thus, identical scaling of u_n^2 and τ_n, and linear decay of correlations cause the largest space scale to contribute to the structure function at arbitrary short time separation t, in the same way as a vortex with eddy turnover time $\tau_n\sim t$, whence the logarithmic correction involving τ_0.

In order to have a quadratic behavior of the time correlation at $t=0$, it is necessary that the noise ξ in Eq. (2.7) be correlated in time, and the correlation must again be given along the trajectories. The appropriate modification to Eq. (2.7) is, therefore,

$$[\partial_t+\mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]q(\mathbf{x},t)+\int d^2y\,\gamma(\mathbf{x}-\mathbf{y})q(\mathbf{y},t)=r(\mathbf{x},t),$$

$$[\partial_t+\mathbf{u}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]r(\mathbf{x},t)+\int d^2y\,\hat{\gamma}(\mathbf{x}-\mathbf{y})r(\mathbf{y},t)$$
$$=\int d^2y\,h(\mathbf{x}-\mathbf{y})\xi(\mathbf{y},t), \tag{3.7}$$

where, for $k \ll \eta^{-1}$,

$$\hat{\gamma}_k = \hat{\rho} C_{Kol}^{1/2} \overline{\epsilon}^{1/3} k^{2/3},$$

$$H_k = |h_k|^2 = 8\pi\rho\hat{\rho}(\rho+\hat{\rho}) C_{Kol}^{5/2} \overline{\epsilon}^{5/3} (k^2+k_0^2)^{2/3}. \quad (3.8)$$

It is easy to show that also the field theory associated with Eq. (3.7) is characterized by marginal interactions: $[\lambda_{\gamma_n}] = [\lambda_{\hat{\gamma}_n}] = [\lambda_{H_n}] = [\lambda_{G_n}] = 0$ and the considerations in Sec. II extend to the present case.

The zero order of the theory leads to the following correlation function:

$$\mathbf{U}_{\mathbf{k}}^L(t) = \frac{\mathbf{k}_\perp \mathbf{k}_\perp}{k^2} \frac{4\pi C_{Kol} \overline{\epsilon}^{2/3}}{(k^2+k_0^2)^{4/3}} \frac{\rho e^{-\hat{\gamma}_k |t|} - \hat{\rho} e^{-\gamma_k |t|}}{\rho - \hat{\rho}} \quad (3.9)$$

and the time correlation has a quadratic maximum at $t=0$. Calculation of the Lagrangian correlation time leads to the same result as of Eq. (3.2), with the substitution $\rho \to \rho\hat{\rho}/(\rho+\hat{\rho})$, while smoothness of the time correlation eliminates the logarithmic correction to the scaling of the Lagrangian time structure function. This structure function obeys, in fact, after sending $k_0 \to 0$, the expected normal diffusion behavior,

$$\langle |\mathbf{u}^L(\mathbf{x},t) - \mathbf{u}^L(\mathbf{x},0)|^2 \rangle = 2 C_0 \overline{\epsilon} |t|,$$

with

$$C_0 = C_{Kol}^{3/2} \frac{\hat{\rho}\rho}{\hat{\rho}-\rho} \ln \hat{\rho}/\rho, \quad (3.10)$$

the constant C_0 is $O(\rho)$ and, as expected from the discussion leading to Eq. (3.6), diverges logarithmically for $\hat{\rho}/\rho \to \infty$.

B. Relative diffusion

Analyzing the transport of a cluster of particles requires consideration of time intervals, during which the space separations involved cannot be approximated as constant. Over these time scales, the short correlation time limit leads to a perturbation scheme, which treats the velocity field to zero order as a white noise.

We focus on the case of a pair of particles. We have to study an equation in the form

$$\begin{aligned} \partial_t [z_\alpha(t|\mathbf{r}_0,0) - z_\alpha(t|0,0)] &= u_\alpha^L(\mathbf{r}_0,t) - u_\alpha^L(0,t) \\ &= U_{\alpha\beta}(\mathbf{z}(t|\mathbf{r}_0,0) - \mathbf{z}(t|0,0)) \xi_\beta(t), \end{aligned} \quad (3.11)$$

with $\langle \xi_\alpha(t)\xi_\beta(0)\rangle = \delta_{\alpha\beta}\delta(t)$ and $U_{\alpha\beta}$ to be determined. Due to the multiplicative noise nature of this equation, attention must be paid to the possible presence of drift terms arising from the Stratonovich prescription implicit in its definition [30]. It is easy to show that this drift is identically zero, either by direct calculation of the increment $\delta\mathbf{z}(t|\mathbf{x},0)$ for t in the inertial range, or noticing that

$$\langle \delta\mathbf{z}(t|\mathbf{r}_0,0)\rangle = \int_0^t d\tau [\langle \mathbf{u}^L(\mathbf{r}_0,\tau)\rangle - \langle \mathbf{u}^L(0,\tau)\rangle] = 0, \quad (3.12)$$

because of homogeneity of turbulence. For this reason, the separation process is described simply by

$$\begin{aligned} &\partial_t \langle [z_\alpha(t|\mathbf{r}_0,0) - z_\alpha(t|0,0)][z_\beta(t|\mathbf{r}_0,0) - z_\beta(t|0,0)]\rangle \\ &\quad = D_{\alpha\beta}(\mathbf{z}(t|\mathbf{r}_0,0) - \mathbf{z}(t|0,0)) \end{aligned} \quad (3.13)$$

with

$$D_{\alpha\beta}(\mathbf{r}) = \int dt \langle [u_\alpha^L(\mathbf{r},t) - u_\alpha^L(0,t)][u_\beta^L(\mathbf{r},0) - u_\beta^L(0,0)]\rangle. \quad (3.14)$$

This tensor is easily calculated from $D_{11}(\mathbf{r})$ for $\mathbf{r}=(r,0)$, exploiting incompressibility. Using $\int_0^{2\pi} d\theta \sin^2\theta \sin^2(x\cos\theta) = (\pi/2)[1 - J_0(2x) - J_2(2x)]$, we find, in the limit $k_0 \to 0$,

$$D_{11}(\mathbf{r}) = \frac{4\alpha_{7/3} C_{Kol}^{1/2} \overline{\epsilon}^{1/3}}{\rho} r^{4/3}, \quad (3.15)$$

where

$$\alpha_{7/3} = \int_0^\infty dx x^{-7/3} [1 - J_0(x) - J_2(x)] \simeq 0.265, \quad (3.16)$$

with J_n the Bessel function of the first kind, is evaluated in terms of gamma functions [31] using the formula $\int_0^\infty dx x^\mu J_\nu(x) = 2^\mu \{\Gamma[\frac{1}{2}(1+\nu+\mu)]/\Gamma[\frac{1}{2}(1+\nu-\mu)]\}$. From incompressibility we find, therefore,

$$D_{\alpha\beta}(\mathbf{r}) = \left[\frac{r_\alpha r_\beta}{r^2} + \frac{7}{3}\left(\delta_{\alpha\beta} - \frac{r_\alpha r_\beta}{r^2}\right)\right] \frac{4\alpha_{7/3} C_{Kol}^{1/2} \overline{\epsilon}^{1/3}}{\rho} r^{4/3}. \quad (3.17)$$

We want to study the asymptotics of the separation process of two particles in the inertial range. The procedure is standard (see, e.g., Ref. [16]); we introduce the distribution P for the separation $\mathbf{r}$ at time t, which will obey the diffusion equation (the summation over repeated indices convention is adopted throughout the paper)

$$\partial_t P = \tfrac{1}{2} \partial_\alpha \partial_\beta D_{\alpha\beta} P \quad (3.18)$$

and look for an isotropic similarity solution in the form

$$P(\mathbf{r},t) = t^{-3} f(t^{-3/2} r) = t^{-3} f(R). \quad (3.19)$$

Equation (3.18) takes then the form

$$\frac{3}{2}\partial_\alpha(R_\alpha f) + \frac{4\alpha_{7/3} C_{Kol}^{1/2} \overline{\epsilon}^{1/3}}{2\rho} \partial_\alpha R^{1/3} R_\alpha \partial_R f = 0. \quad (3.20)$$

This equation has an unphysical solution, which is divergent in $R=0$, and a finite one,

$$f(R)=\exp\left(-\frac{9\rho R^{2/3}}{8\alpha_{7/3}C_{Kol}^{1/2}\overline{\epsilon}^{1/3}}\right), \tag{3.21}$$

whose moments are

$$\langle R^n\rangle=\int_0^\infty R^{1+n}dRf(R)$$
$$=\frac{3}{2}\left(\frac{8\alpha_{7/3}C_{Kol}^{1/2}\overline{\epsilon}^{1/3}}{9\rho}\right)^{3+(3n/2)}\Gamma\left(3+\frac{3n}{2}\right), \tag{3.22}$$

with Γ the standard gamma function.

From here, the expression for the particle space separation is obtained in a straightforward manner; for $\gamma_{x^{-1}}t\gg 1$, indicating $\mathbf{r}(t)=\mathbf{z}(t|\mathbf{r}_0,0)-\mathbf{z}(t|0,0)$,

$$\langle r^2(t)\rangle=c\overline{\epsilon}t^3,\quad c=\frac{10\,240\alpha_{7/3}^3C_{Kol}^{3/2}}{243\rho^3}, \tag{3.23}$$

i.e., the space separation obeys Richardson diffusion. For the relative velocity, we have, from Eq. (2.6),

$$\langle[u_r^L(\mathbf{r}_0,0)-u_r^L(0,0)]^2\rangle=2\alpha^{5/3}C_{kol}\overline{\epsilon}^{2/3}\langle r^{2/3}(t)\rangle, \tag{3.24}$$

where

$$\alpha_{5/3}=\int_0^\infty dxx^{-5/3}[1-J_0(x)-J_2(x)]\simeq 2.149 \tag{3.25}$$

and, using Eq. (3.22), for $\gamma_{x^{-1}}t\gg 1$, we find the normal diffusion behavior,

$$\langle[u_r^L(\mathbf{r}_0,0)-u_r^L(0,0)]^2\rangle=\tilde{c}\overline{\epsilon}t,\quad \tilde{c}=\frac{16\alpha_{5/3}\alpha_{7/3}C_{Kol}^{3/2}}{3\rho}. \tag{3.26}$$

Passing to the smoothed out in time version of the velocity field provided by Eq. (3.7), is accomplished, as in the case of τ_L, by exchanging $\rho\to\rho\hat{\rho}/(\rho+\hat{\rho})$. In Ref. [32], both a subexponential behavior for the function $f(R)$ and Richardson diffusion were observed in a DNS (direct numerical simulation) of two-dimensional turbulence in the inverse cascade regime. Based on the results of that paper, extrapolating applicability of our leading order expressions in ρ would give then (taking also $\hat{\rho}\to\infty$) $\rho\simeq 2$.

C. The role of diffusivity and the Batchelor constant

The dynamics of passive tracers, contrary to that of fluid elements, feels the effect of molecular diffusivity. Due to finiteness of the turbulent correlation times, this effect does not consist purely of an additive noise contribution to the tracer velocity. Indicating by σ the molecular diffusivity, the passive tracer velocity will have the form

$$\mathbf{v}(\mathbf{x},t)+(2\sigma)^{1/2}\boldsymbol{\xi}(\mathbf{x},t) \tag{3.27}$$

with $\langle\xi_\alpha(\mathbf{x},t)\xi_\beta(0,0)\rangle=\delta_{\alpha\beta}\delta(\mathbf{x})\delta(t)$ and $\mathbf{v}$ obeying an equation in the form

$$[\partial_t+\mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]q_v(\mathbf{x},t)+\int d^2y\,\gamma(\mathbf{x}-\mathbf{y})q_v(\mathbf{y},t)$$
$$-\int d^2yh(\mathbf{x}-\mathbf{y})\xi(\mathbf{y},t)$$
$$=-(2\sigma)^{1/2}\langle\boldsymbol{\xi}(\mathbf{x},t)\cdot\boldsymbol{\nabla}q_v(\mathbf{x},t)\rangle_\xi\simeq\sigma\nabla^2q_v(\mathbf{x},t), \tag{3.28}$$

where $q_v=\boldsymbol{\nabla}_\perp\cdot\mathbf{v}$, $\langle\cdot\rangle_\xi$ is an average limited to the noise $\boldsymbol{\xi}$ and use has been made, in converting the advection by molecular noise into a diffusion term, of Itô's lemma [30]. We see (it is assumed that the limit $\eta\to 0$ is already taken) that there is a renormalization of the damping kernel γ,

$$\gamma_k\to\gamma_k+\sigma k^2, \tag{3.29}$$

which leads to a cutoff for the velocity at the inverse diffusive scale,

$$\eta_\sigma^{-1}=(\rho C_{Kol}^{1/2})^{3/4}\overline{\epsilon}^{1/4}\sigma^{-3/4}. \tag{3.30}$$

We have then,

$$\langle\mathbf{v_k}(t)\mathbf{v}_{-\mathbf{k}}(0)\rangle=\frac{\exp(-\sigma k^2|t|)}{1+(k\eta_\sigma)^{4/3}}\langle\mathbf{u_k}(t)\mathbf{u}_{-\mathbf{k}}(0)\rangle, \tag{3.31}$$

and for small space separations $r/\eta_\sigma\to 0$, we have a quadratic behavior for the velocity structure function,

$$\langle[v_r(\mathbf{x}+\mathbf{r},t)-v_r(\mathbf{x},t)]^2\rangle$$
$$=2C_{Kol}\overline{\epsilon}^{2/3}r^2\eta_\sigma^{-4/3}\int_0^\infty\frac{[1-J_0(x)-J_2(x)]dx}{x^{5/3}(x^{4/3}+(r/\eta_\sigma)^{4/3})}$$
$$\simeq\frac{1}{4}C_{Kol}\overline{\epsilon}^{2/3}\eta_\sigma^{-4/3}r^2|\ln r/\eta_\sigma|. \tag{3.32}$$

The transport of a passive scalar $\theta(\mathbf{x},t)$ will be described by the equation

$$[\partial_t+\mathbf{v}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]\theta(\mathbf{x},t)=\sigma\nabla^2\theta(\mathbf{x},t)+f(\mathbf{x},t), \tag{3.33}$$

with $f(\mathbf{x},t)$ a source term. An interesting quantity to calculate is the fluctuation spectrum for θ in the case f is random in time and concentrated at large scale,

$$\langle f(\mathbf{x}+\mathbf{r},t)f(\mathbf{x},0)\rangle=F(r)\delta(t),\quad F(r)=\begin{cases}2\overline{\epsilon}_\theta, & k_0r<1\\ 0, & k_0r>0.\end{cases} \tag{3.34}$$

We can thus consider $\langle\theta\rangle=0$. The equation for the steady state passive scalar correlation $\Theta(r)=\langle\theta(\mathbf{x}+\mathbf{r},t)\theta(\mathbf{x},t)\rangle$ will then be, for $k_0r\ll 1$,

$$\langle \theta(\mathbf{x},t)[\mathbf{v}(\mathbf{x}+\mathbf{r},t)-\mathbf{v}(\mathbf{x},t)]\cdot\nabla\theta(\mathbf{x}+r,t)\rangle$$
$$=2\sigma\nabla^2\Theta(r)+4\overline{\epsilon_\theta}. \tag{3.35}$$

For $r\to 0$, the left hand side of this equation is zero; we thus obtain $\overline{\epsilon_\theta}=(\sigma/2)\langle|\nabla\theta|^2\rangle$, i.e., $\overline{\epsilon_\theta}$ is the dissipation of passive scalar fluctuations. Following the same approach as in the preceding section, the velocity difference $\mathbf{v}(\mathbf{x}+\mathbf{r},t)-\mathbf{v}(\mathbf{x},t)$ is approximated by a white noise. From Itô's lemma, its contribution in Eq. (3.35) will be an eddy diffusivity $D^v_{\alpha\beta}(\mathbf{r})$, whose expression will coincide, for $r\gg\eta_\sigma r$, with the one for $D_{\alpha\beta}$ provided by Eqs. (3.16) and (3.17). Drift terms coming from the Stratonovich prescriptions are ruled out with the arguments used in the preceding section. The resulting diffusion equation will then read,

$$\partial_\alpha\partial_\beta\left(\tfrac{1}{2}D^v_{\alpha\beta}(\mathbf{r})+2\sigma\delta_{\alpha\beta}\right)\Theta(r)+4\overline{\epsilon_\theta}=0. \tag{3.36}$$

For $r\gg\eta_\sigma$ D^v_{ij} is essentially a correction to the molecular diffusivity, and will read, from Eq. (3.31),

$$D^v_\alpha(\mathbf{r})=\int dt\langle[v_\alpha(\mathbf{r},t)-v_\alpha(0,t)][v_\beta(\mathbf{r},0)-v_\beta(0,0)]\rangle$$
$$\simeq\left[\frac{r_\alpha r_\beta}{r^2}+\frac{13}{3}\left(\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)\right]\frac{3\pi\sigma}{8\rho^2}(r/\eta_\sigma)^{10/3}. \tag{3.37}$$

For $\eta_\sigma\ll r$, D^v_{ij} is approximated by Eq. (3.17), the molecular diffusivity σ can be neglected and Eq. (3.35) takes the form

$$\Theta''+\frac{7}{3r}\Theta'=-\frac{8\rho\overline{\epsilon_\theta}}{4\alpha_{7/3}C^{1/2}_{Kol}\bar{\epsilon}^{1/3}r^{4/3}}. \tag{3.38}$$

The solution of this equation gives automatically the passive scalar structure function $\langle[\theta(\mathbf{x}+\mathbf{r},t)-\theta(\mathbf{x},t)]^2\rangle=2[\Theta(0)-\Theta(r)]$ in the inertial range for θ: $\eta_\sigma\ll r\ll k_0^{-1}$. This structure function scales like $r^{2/3}$ and can be written in the form

$$\langle[\theta(\mathbf{x}+\mathbf{r},t)-\theta(\mathbf{x},t)]^2\rangle=\frac{B\overline{\epsilon_\theta}r^{2/3}}{C^{1/2}_{Kol}\bar{\epsilon}^{1/3}}, \tag{3.39}$$

with the parameter $B=3\rho/\alpha_{7/3}$ the so called Batchelor constant of the flow. As with relative diffusion, the case of a velocity field with smooth time correlation described by Eq. (3.7) is recovered substituting ρ with $\rho\hat{\rho}/(\rho+\hat{\rho})$.

IV. SOLID TRACERS: ONE-PARTICLE STATISTICS

We consider the simplest case of a linear drag. In the presence of gravity (or of a constant external force) and of the turbulent velocity field $\mathbf{u}(\mathbf{x},t)$, the solid particle coordinate $\mathbf{z}^P(t|\mathbf{x},0)$ will obey the equation of motion,

$$\dot{\mathbf{z}}^P(t|\mathbf{x},t)=\mathbf{v}^P(\mathbf{x},t)+\mathbf{u}_G,\quad \mathbf{z}^P(0|\mathbf{x},0)=\mathbf{x}, \tag{4.1}$$

where $\mathbf{u}_G$ is the gravitational drift that we suppose constant and uniform and $\mathbf{v}^P$ is the fluctuation in the Lagrangian solid particle velocity, which obeys the linear relaxation equation

$$\dot{\mathbf{v}}^P(\mathbf{x},t)=\tau_S^{-1}[\mathbf{u}(\mathbf{z}^P(t|\mathbf{x},0),t)-\mathbf{v}^P(\mathbf{x},t)]$$
$$=\tau_S^{-1}[\mathbf{u}^P(\mathbf{x},t)-\mathbf{v}^P(\mathbf{x},t)], \tag{4.2}$$

with τ_S the Stokes time. (For a spherical particle of radius a and density ρ_P, in a fluid of density ρ_0 and kinematic viscosity ν, we would have: $(2a^2/9\nu)|1-\rho_P/\rho_0|$; we are disregarding any effect from finite particle Reynolds number [33].) From now on we shall identify Lagrangian quantities calculated on solid particle trajectories by the superscript P.

In general the noncoincidence of fluid and solid particle trajectories makes the analysis of Eqs. (4.1) and (4.2) a very difficult task. The short correlation time limit $\rho\to\infty$, however, allows us to proceed perturbatively in the fluctuating part of the trajectory separation $\mathbf{u}_Gt+\mathbf{z}(t|\mathbf{x},0)-\mathbf{z}^P(t|\mathbf{x},0)$. The physical motivation for this is that, from Eq. (4.2), $\mathbf{u}_Gt+\mathbf{z}(t|\mathbf{x},0)-\mathbf{z}^P(t|\mathbf{x},0)$ fluctuates on time scale τ_S with velocity scale fixed by those eddies that have decay time τ_S. Hence, for ρ large, the fluctuating part of trajectory separation remains small on the scale of these eddies. Furthermore, when either $u_Gt>\delta z(t|\mathbf{x}+\mathbf{u}_Gt,\mathbf{x})$, or $\gamma_{|u_Gt|^{-1}}t>1$, in other words, when either $C^{3/2}_{Kol}\bar{\epsilon}t/u_G^2<1$ or $C^{3/2}_{Kol}\bar{\epsilon}t/u_G^2>\rho^{-3}$ (provided $\rho>1$, one of the two conditions is always satisfied), it is possible to approximate $\mathbf{z}(t|\mathbf{x},0)+\mathbf{u}_Gt\simeq\mathbf{z}(t|\mathbf{x}+\mathbf{u}_Gt,0)$.

To lowest order we have, therefore,

$$\mathbf{u}(\mathbf{z}^P(t|\mathbf{x},0),t)=\mathbf{u}(\mathbf{u}_Gt+\mathbf{z}(t|\mathbf{x},0),t)=\mathbf{u}^L(\mathbf{x}+\mathbf{u}_Gt,t). \tag{4.3}$$

We obtain immediately the fluctuation amplitude of the velocity difference between solid and fluid particles at a given position. From Eqs. (4.2) and (4.3) we can write,

$$\mathbf{v}^P(\mathbf{x},t)=\int\frac{d^2k}{(2\pi)^2}\int_{-\infty}^t\frac{d\tau}{\tau_S}\mathbf{u}^L_{\mathbf{k}}(\tau)\exp\left(-\frac{t-\tau}{\tau_S}+i\mathbf{k}\cdot\mathbf{x}\right) \tag{4.4}$$

and from here we obtain, using Eq. (3.2),

$$\langle(v_\alpha-u_\alpha)(v_\beta-u_\beta)\rangle$$
$$=\delta_{\alpha\beta}u_S^2\int_1^\infty\frac{dx}{x\left(1+\dfrac{2\tau_S}{\tau_L}x^{1/3}\right)}\xrightarrow[\tau_S\ll\tau_L]{}3\delta_{\alpha\beta}u_S^2\ln(\tau_L/\tau_S), \tag{4.5}$$

where

$$u_S=\left(\frac{\tau_S}{3\tau_L}\right)^{1/2}u_T \tag{4.6}$$

for $\tau_S<\tau_L$, is the velocity scale of eddies with lifetime τ_S and u_T is the turbulent velocity defined in Eq. (3.3). In order to proceed to next order, it is necessary to calculate the trajectory separation,

$$\mathbf{z}^P(t|\mathbf{x},0)-\mathbf{z}(t|\mathbf{x},0)=\mathbf{u}_G t+(1-e^{-t/\tau_S})\int_{-\infty}^0 d\tau\ \ e^{\tau/\tau_S}\mathbf{u}^L(\mathbf{x},\tau)$$
$$-\int_0^t d\tau \exp\left(-\frac{t-\tau}{\tau_S}\right)\mathbf{u}^L(\mathbf{x},\tau). \qquad (4.7)$$

We notice from this equation that the inertia produced part of trajectory separation does not grow indefinitely. In other words, if $\mathbf{u}_G=0$ and to lowest order in ρ^{-1}, there will be localization of solid particle trajectories around the fluid parcel trajectories they cross at any given time; from Eq. (4.7): $\langle|\mathbf{z}^P(t|\mathbf{x},-\infty)-\mathbf{z}(t|\mathbf{x},-\infty)|^2\rangle\sim(u_T\tau_S)^2\sim C_{Kol}\overline{\epsilon^{2/3}}k_0^{-2/3}\tau_S^2$. We thus introduce the localization length S_l,

$$S_l=C_{Kol}^{1/2}\overline{\epsilon^{1/3}}k_0^{-1/3}\tau_S. \qquad (4.8)$$

What happens is that the velocity difference $\mathbf{v}^P-\mathbf{u}^P$ obeys a relaxation equation with a forcing that is a time derivative; from Eq. (4.2): $(d/dt)(\mathbf{v}^P-\mathbf{u}^P)+\tau_S^{-1}(\mathbf{v}^P-\mathbf{u}^P)=-\dot{\mathbf{u}}^P$. The frequency spectrum of $\mathbf{v}^P-\mathbf{u}^P$ does not have, therefore, the small frequency singularity necessary for long time divergence. The localization length S_l will appear to play a fundamental role in the production both of concentration fluctuations and of corrections to the velocity correlation time. (Of course, to higher order in ρ^{-1}, the relative separation of fluid parcels sets in and localization is destroyed; S_l becomes then, that part of trajectory separation which remains after the Richardson diffusion contribution is subtracted out.)

In the absence of gravity, beside the integral scale dependent localization length S_l, three more scales, which, if $\tau_S \ll \tau_L$, are purely inertial, can be obtained combining τ_S, the crossing time of an eddy by a solid particle, the eddy lifetime, and the eddy turnover time. We have the size S of an eddy whose lifetime equals τ_S, $\gamma_{S^{-1}}\tau_S\sim 1$; the size S_c of an eddy that is crossed by a solid particle in a time τ_S, $u_S \sim S_c/\tau_S$; the size S_i of an eddy whose lifetime equals the crossing time by a solid particle, $S_i\gamma_{S_i^{-1}}\sim u_S$. Summarizing,

$$S=\rho^{3/2}C_{Kol}^{3/4}\overline{\epsilon^{1/2}}\tau_S^{3/2}, \quad S_c=\rho^{1/2}C_{Kol}^{3/4}\overline{\epsilon^{1/2}}\tau_S^{3/2},$$
$$S_i=\rho^{-3/2}C_{Kol}^{3/4}\overline{\epsilon^{1/2}}\tau_S^{3/2}. \qquad (4.9)$$

From Eq. (4.9), we identify the following sequence of ranges.

A large separation range $r>S$, in which the fluid velocity $\mathbf{u}^P$ varies slowly on the scale of the relaxation time τ_S.

A first intermediate range $S<r<S_c$ in which the fluid velocity $\mathbf{u}^P$ is a fast variable, but still, τ_S is short compared with the crossing time of an eddy of size r; hence, Eq. (4.2) has the form of a Langevin equation with a noise $\tau_S^{-1}\mathbf{u}^P$ of constant amplitude on the scale of this crossing time. The crossover scale S will play an important role in the determination of the degree of nonergodicity of the solid particle flow (see Sec. VI).

A second intermediate range $S_c<r<S_i$, in which the crossing time is shorter than both the Stokes time and the eddy turnover time, but is longer than the lifetime of an eddy of that size; hence, the solid particle moves ballistically with respect to the fluid.

A small separation range $r<S_i$, in which trajectory separation in the lifetime of an eddy is not a perturbation anymore.

From Eqs. (4.3), (4.4), and (4.7), we can establish a perturbative calculation scheme for $\mathbf{u}^P$ and $\mathbf{v}^P$. Notice that, within perturbation theory, $\mathbf{v}^P$ is a one-valued function of $\mathbf{x}$ and t, and $\mathbf{v}(\mathbf{x},t)$ defines automatically a velocity field for the solid particles. The separation between S_i and all the other scales of the problem, has the consequence that, in the present case, the Weinstock approximation is exact [34]. What happens is that trajectory separation is produced mainly by eddies of size $r\gtrsim S$, for which trajectory separation is a perturbation. This has the consequence, in particular, that the Weinstock approximation applies also at scales $r<S_i$ for which trajectory separation is not a perturbation at all. For dominant gravity, i.e., when $u_G>u_S$, trajectory separation is produced mainly by the gravitational drift u_G and the Weinstock approximation is automatically satisfied.

We can calculate at this point the time correlation for the solid particle velocity and adopt the approach followed in Refs. [35,36]; we can thus write, using Eq. (4.7),

$$\langle u_1^P(0,0)u_1^P(0,t)\rangle$$
$$=\int \frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\langle u_{1\mathbf{k}}^L(0)u_{1\mathbf{p}}^L(t)$$
$$\times\exp[i\mathbf{p}\cdot(\mathbf{z}^P(t|0,0)-\mathbf{z}(t|0,0))]\rangle$$
$$=-\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\exp(i\mathbf{p}\cdot\mathbf{u}_G t)\left.\frac{\delta^2 Z[\mathbf{J}]}{\delta J_{\mathbf{k}1}(0)\,\delta J_{\mathbf{p}1}(t)}\right|_{\mathbf{J}=\mathbf{p}\bar{J}_t}, \qquad (4.10)$$

where

$$Z[\mathbf{J}]=\left\langle\exp\left(i\int\frac{d^2s}{(2\pi)^2}\int dt\mathbf{u}_{\mathbf{s}}^L(t)\cdot\mathbf{J}_{\mathbf{s}}(t)\right)\right\rangle$$
$$=\mathcal{N}\exp\left(-\frac{1}{2}\int d\tau d\tau'\int\frac{d^2s}{(2\pi)^2}\right.$$
$$\left.\times\mathbf{J}_{\mathbf{s}}(\tau)\cdot\mathbf{U}_{\mathbf{s}}^L(\tau-\tau')\cdot\mathbf{J}_{-\mathbf{s}}(\tau')\right) \qquad (4.11)$$

is the generating functional for the field $\mathbf{u}^L$ and

$$\bar{J}_t(\tau)=\begin{cases}0, & \tau>t\\ -\exp\left(-\dfrac{t-\tau}{\tau_S}\right), & 0<\tau<t\\ [1-\exp(-t/\tau_S)]\exp(\tau/\tau_S), & \tau<0.\end{cases} \qquad (4.12)$$

Substituting back into Eq. (4.10), we obtain, after introducing dimensionless variables $\bar{t}=t/\tau_S$, $\bar{u}_G=k_0\tau_S u_G$, and $\bar{\gamma}=\tau_S\gamma_{k_0}=\tau_S/(2\tau_L)$,

$$\langle u_1^P(0,0)u_1^P(0,t)\rangle = \frac{u_T^2}{6}\int_1^\infty dx x^{-4/3}[J_0(\bar{u}_G(x-1)^{1/2}\bar{t})+J_2(\bar{u}_G(x-1)^{1/2}\bar{t})]$$
$$\times\exp\left[-\bar{\gamma}\bar{t}x^{1/3}-\frac{\bar{\gamma}^2(x-1)}{2\rho^2}\int_1^\infty \frac{dy}{y^{4/3}(1+\bar{\gamma}y^{1/3})}\left(1-e^{-\bar{t}}-\frac{e^{-\bar{\gamma}\bar{t}y^{1/3}}-e^{-\bar{t}}}{1-\bar{\gamma}y^{1/3}}\right)\right]. \quad (4.13)$$

We see from this equation that decorrelation of the fluid velocity sampled by a solid particle receives three contributions: one from the gravitational drift u_G, one from the eddy decay $\bar{\gamma}x^{1/3}\bar{t}$, and the integral term in the exponential, which comes from inertia produced trajectory separation. This last term is peculiar, in that it saturates to a constant for long t instead of continuing to increase indefinitely. This term is the argument in the exponential expression for $Z[\mathbf{J}]$ [see Eq. (4.11)], which is essentially

$$\mathbf{pp}:\langle[\mathbf{z}^P(t|0,0)-\mathbf{z}(t|0,0)][\mathbf{z}^P(t|0,0)-\mathbf{z}(t|0,0)]\rangle, \quad (4.14)$$

with the drift $\mathbf{u}_G$ subtracted out, and with $\mathbf{p}$ the wave vector entering the integral of Eq. (4.10). But, from Eqs. (4.7) and (4.8), we saw that this expression saturates at $t\to\infty$. In consequence of this, for long enough times, the large x behavior of the integrand in Eq. (4.13) will be dominated by the value at saturation of the inertia produced term.

A. Velocity self-diffusion

Inertia causes two ranges of time separations in the correlation $\langle u_1^P(\mathbf{x},0)u_1^P(\mathbf{x},t)\rangle$: one at short times dominated by sweep from the velocity difference $\mathbf{u}_G+\mathbf{v}-\mathbf{u}$ and one at long time associated with eddy decay, where Eq. (3.10) holds [22]. The transition between the two ranges occurs at

$$t\sim\frac{\max(u_G^2,u_S^2)}{\rho^3 C_{Kol}^{3/2}\bar{\epsilon}}. \quad (4.15)$$

From Eqs. (4.5) and (4.6), for dominant inertia, i.e., $u_S > u_G$, this crossover time is much shorter than τ_L, while, for dominant gravity, i.e., for $u_G \gg u_S$ it is possible that sweep dominates for all inertial time scales; for this to occur, it is necessary that the crossing time of a large eddy by the particle be less than τ_L, i.e., $k_0 u_G \tau_L > 1$. For dominant inertia the crossover time $u_S^2/(\rho^3 C_{Kol}^{3/2}\bar{\epsilon})\sim\rho^{-2}\tau_S$ is just the lifetime of an eddy of size S_i [see Eq. (4.9)].

For dominant gravity, the exponential term in Eq. (4.13) can be neglected. For $t\ll\min(\tau_G,\tau_L)$ with $\tau_G=(6/\rho^2)(u_G/u_T)^2\tau_L\sim u_G^2/\rho^3 C_{Kol}^{3/2}\bar{\epsilon}$, we find

$$\langle[u_1^P(0,t)-u_1^P(0,0)]^2\rangle = \frac{u_T^2}{3}\int_1^\infty dx x^{-4/3}[1-J_0(\bar{u}_G\bar{t}(x-1)^{1/2})-J_2(\bar{u}_G\bar{t}(x-1))^{1/2}]\simeq\frac{2}{3}\alpha_{5/3}C_{Kol}\bar{\epsilon}^{2/3}(u_G t)^{2/3}, \quad (4.16)$$

where $\alpha_{5/3}\simeq 2.149$ [see Eq. (3.25)]. The time τ_G, for $u_G < u_L$, is the lifetime of vortices whose lifetime equals the crossing time by a falling particle; for $\rho=O(1)$, τ_G coincides with the eddy turnover time of vortices with characteristic velocity u_G.

For dominant inertia and short enough times, only the last piece in Eq. (4.13) will contribute and will be quadratic in $\bar{t}$; if $\tau_S\ll\tau_L$,

$$1-e^{-\bar{t}}-\frac{e^{-\bar{\gamma}\bar{t}y^{1/3}}-e^{-\bar{t}}}{1-\bar{\gamma}y^{1/3}}\simeq\frac{1}{2}\bar{\gamma}y^{1/3}\bar{t}^2. \quad (4.17)$$

Substituting into Eq. (4.13), we are left with the following expression:

$$\langle[u_1^P(0,t)-u_1^P(0,0)]^2\rangle = \frac{u_T^2}{3}\int_1^\infty dx x^{-4/3}\left\{1-\exp\left[-\frac{\bar{\gamma}^3\bar{t}^2(x-1)}{4\rho^2}\int_1^\infty\frac{dy}{y(1+\bar{\gamma}y^{1/3})}\right]\right\}$$
$$\simeq\frac{u_T^2}{3}\int_0^\infty dx x^{-4/3}\left\{1-\exp\left[\frac{3\bar{\gamma}^3\bar{t}^2 x\ln\bar{\gamma}}{4\rho^2}\right]\right\}. \quad (4.18)$$

Using $\int_0^\infty dx x^{-4/3}[1-\exp(-Ax)]=3\Gamma(2/3)A^{1/3}$, we obtain, therefore,

$$\langle[u_1^P(0,t)-u_1^P(0,0)]^2\rangle \simeq \tfrac{3}{2}\Gamma(2/3)[3\ln(\tau_L/\tau_S)]^{1/3} \times C_{Kol}\overline{\epsilon}^{2/3}(u_S t)^{2/3}. \qquad (4.19)$$

Comparing with Eqs. (3.10) and (3.14), we see that inertia will dominate if $u_S > u_G$ and $t < \rho^{-2}\tau_S$. As predicted in Ref. [22], at short times, the time structure function for $\mathbf{u}^P$ has a subdiffusive behavior with exponent 2/3 both for dominant u_G and dominant u_S. What happens is that at such short time scales, the particle crosses at constant speed (remember also, in the inertia dominated case, that $S_c \gg S_i$) vortices whose velocity field is, in the limit, basically frozen; hence a Taylor hypothesis applies, and time correlations coincide with their spatial counterparts.

B. Velocity correlation times

Starting from Eq. (4.13), we can calculate the correlation time τ_P for the fluid velocity sampled by a solid particle,

$$\tau_P = \langle[u_1^P]^2\rangle^{-1}\int_0^\infty dt\langle u_1^P(0,0)u_1^P(0,t)\rangle. \qquad (4.20)$$

To lowest order, any discrepancy between the PDFs (probability distribution functions) for u^L and u^P can be neglected and we have $\langle[u_1^P]^2\rangle = \langle[u_1^L]^2\rangle = \frac{1}{2}u_T^2$. We begin by analyzing the case of dominant inertia: $u_G = 0$. Taylor expanding in ρ^{-1} the integrand in Eq. (4.13) and substituting into Eq. (4.20), leads to terms that diverge when integrated in x. This indicates that the time independent part of the inertia term in Eq. (4.13) dominates the integral. We thus Taylor expand in ρ^{-1}, only the time dependent piece of the integrand in Eq. (4.13), i.e.,

$$\exp\left[-\overline{\gamma t}x^{1/3} + \frac{\overline{\gamma}^2(x-1)}{2\rho^2}\int_1^\infty \frac{dy}{y^{4/3}(1+\overline{\gamma}y^{1/3})} \times\left(e^{-\overline{t}} + \frac{e^{-\overline{\gamma t}y^{1/3}} - e^{-\overline{t}}}{1-\overline{\gamma}y^{1/3}}\right)\right] \qquad (4.21)$$

to obtain

$$\int_0^\infty dt\langle u_1^P(\mathbf{x},0)u_1^P(\mathbf{x},t)\rangle = \frac{u_T^2}{6}\int_1^\infty dx x^{-4/3}\exp\left(-\frac{\overline{\gamma}^2(x-1)}{2\rho^2}\int_1^\infty\frac{dy}{y^{4/3}(1+\overline{\gamma}y^{1/3})}\right) \times\left[\frac{1}{\overline{\gamma}x^{1/3}} + \frac{\overline{\gamma}(x-1)}{2\rho^2}\int_1^\infty dy\frac{1+\overline{\gamma}x^{1/3}-\overline{\gamma}^2y^{1/3}(x^{1/3}+y^{1/3})}{y^{4/3}(1-\overline{\gamma}^2y^{2/3})(1+\overline{\gamma}x^{1/3})(x^{1/3}+y^{1/3})}\right] \qquad (4.22)$$

and we see that the integral in x of the $O(\rho^{-2})$ on second line of Eq. (4.22) is dominated in fact by a saddle point at $x = (k/k_0)^2 \sim (\rho/\overline{\gamma})^2$, i.e., at $k \sim S_l^{-1}$. Combining this result, with the fact that the integrands are peaked at $y \sim 1$, Eq. (4.22) will take the form

$$\int_0^\infty dt\langle u_1^P(0,0)u_1^P(0,t)\rangle = \frac{u_T^2}{6}\int_1^\infty dx x^{-4/3}\exp\left(-\frac{\overline{\gamma}^2x}{2\rho^2}\int_1^\infty\frac{dy}{y^{4/3}(1+\overline{\gamma}y^{1/3})}\right)\left[\frac{1}{\overline{\gamma}x^{1/3}} + \frac{\overline{\gamma}x^{2/3}}{2\rho^2}\int_1^\infty\frac{dy}{y^{4/3}(1+\overline{\gamma}y^{1/3})}\right]. \qquad (4.23)$$

We thus obtain, for the deviation $\tau_P - \tau_L$,

$$\frac{\tau_P}{\tau_L} = 1 + B(\overline{\gamma})\overline{\gamma}^{4/3}\rho^{-4/3} + O(\rho^{-2}), \qquad (4.24)$$

where

$$B(\overline{\gamma}) = \left(\frac{2}{3}\right)^{1/3}\Gamma(1/3)\left[\frac{1}{3} - \frac{\overline{\gamma}}{2} + \overline{\gamma}^2 + \overline{\gamma}^3\ln\frac{\overline{\gamma}}{1+\overline{\gamma}}\right]^{2/3}. \qquad (4.25)$$

It is to be noticed that the factor $B(\overline{\gamma})$ is always positive, i.e., the correlation time for the fluid velocity seen by the solid particle is longer than τ_L. Following the argument in Ref. [37], this would be expected in the case of a velocity field with statistics defined in an Eulerian frame, and is exactly the result obtained in Ref. [38]. In the case of a Lagrangian statistics, it is not clear whether the deviation between solid and fluid particle trajectories should have led to a faster, rather than slower, decorrelation rate.

In the case of dominant gravity, as expected [17,36], there is always a decrease of the correlation time. In place of Eq. (4.22), we have

$$\int_0^\infty dt\langle u_1^P(0,0)u_1^P(0,t)\rangle = \frac{u_T^2}{6}\int_0^\infty dt\int_1^\infty dx x^{-4/3}[J_0(\overline{u_G}(x-1)^{1/2}\overline{t}) + J_2(\overline{u_G}(x-1)^{1/2}\overline{t})]\exp(-\overline{\gamma t}x^{1/3}), \qquad (4.26)$$

which, using $\int_0^\infty dx J_\nu(\beta x)e^{-\alpha x} = \beta^{-\nu}(\alpha^2+\beta^2)^{-1/2}[(\alpha^2+\beta^2)^{1/2}-\alpha]^\nu$ [31], leads to the expression for the correlation time,

$$\frac{\tau_P}{\tau_L}=\frac{2}{3}\int_0^\infty \frac{dx}{x^{4/3}}\left(\frac{\overline{\gamma}x^{1/3}}{\overline{u}_G(x-1)^{1/2}}\right)^2\left\{\left[\left(\frac{\overline{\gamma}x^{1/3}}{\overline{u}_G(x-1)^{1/2}}\right)^2+1\right]^{1/2}\right\}. \tag{4.27}$$

We can obtain limiting expressions for this ratio, when the crossing time $(k_0u_G)^{-1}$ is much longer or much shorter than the integral time τ_L,

$$\frac{\tau_P}{\tau_L}=\begin{cases}1+\frac{2}{3}(u_Gk_0\tau_L)^2\ln u_Gk_0\tau_L, & k_0u_G\tau_L\ll 1\\ 2^{3/2}(u_Gk_0\tau_L)^{-1}, & k_0u_G\tau_L\gg 1.\end{cases} \tag{4.28}$$

C. Eulerian correlations

The limit $\tau_S\to\infty$, corresponding to the case of a particle with infinite inertia, leads, from Eq. (4.2), to a particle velocity, which, in the absence of gravity, is identically zero. Hence $\mathbf{u}^P(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)$ and the time statistics for the fluid velocity seen by the particle coincides with the Eulerian turbulent statistics. In this regime, the dimensionless units introduced for Eq. (4.13) are not appropriate anymore. Redefining $\overline{t}=\gamma_{k_0}t$, Eq. (4.13) takes the following form, after writing $\exp(-t/\tau_S)\simeq 1-t/\tau_S$:

$$\langle u_1(0,0)u_1(0,t)\rangle=\frac{u_T^2}{6}\int_1^\infty dx x^{-4/3}\exp\left[-\overline{t}x^{1/3}-\frac{(x-1)}{2\rho^2}\times\left(\frac{3\overline{t}}{2}-1+\int_1^\infty dy y^{-2}\exp(-\overline{t}y^{1/3})\right)\right]. \tag{4.29}$$

We start by calculating the Eulerian correlation time

$$\tau_E=u_T^{-2}\int dt\langle\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},0)\rangle. \tag{4.30}$$

Contrary to Eq. (4.13), it is the linear in t, $O(\rho^{-2})$ term in Eq. (4.29), which, at fixed long enough t, dominates for $x\to\infty$. The same reasons leading to expand Eq. (4.21) suggest that we must now expand

$$\exp\left[\frac{(x-1)}{2\rho^2}\left(1-\int_1^\infty dy y^{-2}\exp(-\overline{t}y^{1/3})\right)\right]. \tag{4.31}$$

Instead of Eq. (4.22), we find

$$\int_0^\infty dt\langle u_1(0,0)u_1(0,t)\rangle=\frac{u_T^2}{6}\int_1^\infty dx x^{-4/3}\left[\left(1+\frac{(x-1)^2}{2\rho^2}\right)\left(x^{1/3}+\frac{3(x-1)}{4\rho^2}\right)^{-1}-\frac{(x-1)}{2\rho^2}\int_1^\infty dy y^{-2}\left(x^{1/3}+y^{2/3}+\frac{3(x-1)}{4\rho^2}\right)^{-1}\right]. \tag{4.32}$$

All the terms involving factors ρ^{-2} lead, after integration, to an $O(\rho^{-2})$ result, except one that leads to an $O(\rho^{-2}\ln\rho)$ term; the integral in Eq. (4.32) will read, to leading order in ρ,

$$\int_1^\infty dx\,[x^{-5/3}-\rho^{-2}x^{-1}(1+\rho^{-2}x^{2/3})^{-1}]+O(\rho^{-2})\simeq\frac{3}{2}-\frac{3\ln\rho}{\rho^2}. \tag{4.33}$$

We obtain then the result for the Eulerian correlation time,

$$\frac{\tau_E}{\tau_L}=1-\frac{2\ln\rho}{\rho^2}, \tag{4.34}$$

which is shorter than τ_L, as expected from the fact that the velocity field statistics is defined along fluid trajectories, and, sampling at fixed space position should lead to an increase in the rate of decorrelation. Comparing Eqs. (4.28) and (4.36), we see therefore that there is a transition from a correlation time longer than τ_L for light particles, to a shorter one for heavy particles. The origin of this lies in the opposite orderings $\tau_S\lesssim t$ and $\tau_S\gg t$, on which the Taylor expansions of Eqs. (4.21) and (4.31) are based. [More precisely, for $\tau_S>\rho\tau_L$, we have $k_0S_l>1$ and the saddle point in Eq. (4.22) disappears.]

As a last exercise, it is possible to calculate the sweep produced decay in an Eulerian two-point two-time structure function in the form

$$S_{rr}(r,t)=\langle[u_r(\mathbf{r},t)-u_r(0,t)][u_r(\mathbf{r},0)-u_r(0,0)]\rangle. \tag{4.35}$$

From the discussion leading from Eq. (3.14) to Eq. (3.17), one finds that the structure function in Eq. (4.35) is obtained by inserting a factor $2[1-J_0(rx)-J_2(rx)]$ in the integrand of Eq. (4.29). If one considers shorter time and space scales $k_0r\ll 1$, $t\ll\tau_L$, the leading cause of correlation decay is sweep, and the $\overline{t}x^{1/3}$ in the integrand of Eq. (4.29) can be disregarded. Again because of shortness of t/τ_L, one can Taylor expand $\exp(-\overline{t}y^{1/3})$ in the same equation and the final result is

$$S_{rr}(r,t)\simeq 2C_{Kol}\bar{\epsilon}^{2/3}r^{2/3}\int_0^\infty dx x^{-5/3}[1-J_0(x)-J_2(x)]\times\exp\left[-\frac{u_T^2t^2x^2}{6r^2}\right]. \tag{4.36}$$

The term in the exponent is $O(t/T_{r^{-1}})^2$, with $T_{r^{-1}}$ the sweep time at scale r. Hence, if $t\gg T_{r^{-1}}$, it is possible to Taylor expand the Bessel functions and the result is

$$S_{rr}(r,t)\sim S_{rr}(r,0)\int_0^\infty dx x^{1/3}\exp[-(t/T_{r^{-1}})^2x^2]\sim S_{rr}(r,0)\left(\frac{T_{r^{-1}}}{t}\right)^{4/3}, \tag{4.37}$$

i.e., a power-law decay of the structure function for times longer than the sweep time at that space separation.

V. SOLID TRACERS: CONCENTRATION FLUCTUATIONS

Because of inertia, the particle velocity field $\mathbf{v}(\mathbf{x},t)$, contrary to $\mathbf{u}(\mathbf{x},t)$, does not preserve volume. Physical intuition suggests that particles that are denser than the fluid, will tend to concentrate near the instantaneous hyperbolic points of the flow, and to escape from the elliptic ones [19,39]. For this reason, a distribution $\theta(\mathbf{x},t)$ of solid particles, in the absence of external sources, will be characterized by finite amplitude fluctuations superimposed to a uniform mean concentration field $\bar{\theta}$. These fluctuations are expected to have a correlation time of the order of τ_S and a correlation length determined in consequence. We are going to neglect any effect of gravity and set from the start $\mathbf{u}_G=0$. We will also limit our analysis to the case in which τ_S is in the turbulent inertial range, i.e., we consider $\tau_S\ll\tau_L$ (more precisely, $\tau_S<\rho^{-2}\tau_L$). In this way, all nonuniversal effects associated with the large scales of the flow are eliminated from the problem.

The length S_i is crucial to the two-particle statistics, in that it gives the scale below which solid particles move ballistically relative to one another. In fact, S_c fixes the crossover scale to ballistic behavior, only for the relative motion of solid and fluid particles; the resulting picture is given by pairs of particles, separated by S_i, moving ballistically over scale S_c. It is easy to see this: if $\Delta_r v$ is the typical relative velocity between two solid particles at separation r and $\Delta_r u\sim C_{Kol}^{1/2}(\bar{\epsilon}r)^{1/3}$ is the corresponding value for the fluid velocity, one will have for $r\ll S$, from Eq. (4.2): $\Delta_r v\sim(\tau_S\gamma_{r^{-1}})^{-1/2}\Delta_r u$; exploiting the fact that the characteristic time of variation for v is τ_S, the condition $\tau_S\Delta_r v\sim r$, gives then $r\sim S_i$.

The concentration correlation $\Theta(\mathbf{r})=\langle\theta(\mathbf{r},t)\theta(0,t)\rangle$ is proportional to the equilibrium PDF $P(\mathbf{r})$ for the separation of a pair of solid particles advected by $\mathbf{u}(\mathbf{x},t)$. The separation $\mathbf{r}(t)$ obeys an equation in the form $\dot{\mathbf{r}}(t)=\mathbf{v}^P(\mathbf{x}+\mathbf{r},t)-\mathbf{v}^P(\mathbf{x},t)$ [we use from now on the shorthand $\mathbf{r}(t)\equiv\delta\mathbf{z}^P(t|\mathbf{x}+\mathbf{r},0)$], and, for $r\gg S_i$, the separation process takes a diffusive nature,

$$\frac{d}{dt}\langle[r_\alpha(t)-r_\alpha(0)][r_\beta(t)-r_\beta(0)]\rangle=2D_{\alpha\beta}(\mathbf{r}). \tag{5.1}$$

A finite level of concentration fluctuations, in the absence of external sources, is associated with a finite divergence of the diffusivity tensor: $\partial_\alpha D_{\alpha\beta}\neq 0$. If this component of the diffusivity tensor is small, it is possible to proceed perturbatively: $D_{\alpha\beta}=D_{\alpha\beta}^{(0)}+D_{\alpha\beta}^{(1)}$, $P=P^{(0)}+P^{(1)}$, with $\partial_\alpha D_{\alpha\beta}^{(0)}=0$, $P^{(0)}$ uniform and $P^{(1)}(\mathbf{r})\propto\langle[\theta(\mathbf{r},t)-\theta(0,t]^2\rangle$; the equation for the fluctuation amplitude $P^{(1)}(\mathbf{r})$ would read, therefore,

$$D_{\alpha\beta}^{(0)}\partial_\alpha\partial_\beta P^{(1)}=-P^{(0)}\partial_\alpha\partial_\beta D_{\alpha\beta}^{(1)}. \tag{5.2}$$

The procedure to determine $D_{\alpha\beta}$ is similar to the one leading to Eq. (3.17). From Eq. (4.4) and the relation $\dot{\mathbf{r}}(t)=\mathbf{v}^P(\mathbf{x}+\mathbf{r},t)-\mathbf{v}^P(\mathbf{x},t)$, we obtain

$$D_{\alpha\beta}(\mathbf{r})=\lim_{T\to\infty}\frac{1}{T}\int_0^T dt_1\int_0^T dt_2\int_{-\infty}^{t_1}\frac{d\tau_1}{\tau_S}\int_{-\infty}^{t_2}\frac{d\tau_2}{\tau_S}\times\exp\left(-\frac{t_1+t_2-\tau_1-\tau_2}{\tau_S}\right)S_{\alpha\beta}^P(\mathbf{r},\tau_1,\tau_2) \tag{5.3}$$

with $S_{\alpha\beta}^P$ the time correlation of velocity differences along solid particle trajectories,

$$S_{\alpha\beta}^P(\mathbf{r},t_1,t_2)=\langle[u_\alpha^P(\mathbf{r},t_1)-u_\alpha^P(0,t_1)][u_\beta^P(\mathbf{r},t_2)-u_\beta^P(0,t_2)]\rangle=2[\langle u_\alpha^P(\mathbf{r},t_1)u_\beta^P(\mathbf{r},t_2)\rangle-\langle u_\alpha^P(\mathbf{r},t_1)u_\beta^P(0,t_2)\rangle]. \tag{5.4}$$

We notice that, if we approximated $S_{\alpha\beta}^P(\mathbf{r},t_1,t_2)=S_{\alpha\beta}^L(\mathbf{r},t_1-t_2)$, since $\partial_\alpha S_{\alpha\beta}^L(\mathbf{r},t_1-t_2)=0$, we would obtain from Eq. (5.3) a divergenceless $D_{\alpha\beta}(\mathbf{r})$. We have to take into account therefore the effect of trajectory separation described in Eq. (4.7). Proceeding as in the case of the one-particle statistics, we arrive at the following modification of Eq. (4.10):

$$\langle u_\alpha^P(0,t_1)u_\beta^P(\mathbf{r},t_2)\rangle=-\int\frac{d^2k}{(2\pi)^2}\frac{d^2p}{(2\pi)^2}\exp(i\mathbf{p}\cdot\mathbf{r})\times\left.\frac{\delta^2Z[\mathbf{J}]}{\delta J_{\mathbf{k}\alpha}(t_1)\delta J_{\mathbf{p}\beta}(t_2)}\right|_{\mathbf{J}=\mathbf{p}\tilde{J}_{\mathbf{r}t_1t_2}}, \tag{5.5}$$

where

$$\tilde{J}_{\mathbf{s},\mathbf{r}t_1t_2}(\tau)=\overline{J}_{t_1}(\tau)-e^{i\mathbf{s}\cdot\mathbf{r}}\overline{J}_{t_2}(\tau) \tag{5.6}$$

and $Z[\mathbf{J}]$ and $\overline{J}_t$ are given in Eqs. (4.10) and (4.11). Carrying out the wave vector and time integrations in the definition of $Z[\mathbf{J}]$ and using Eqs. (5.6) and (3.1) leads, after some algebra, to the following expression for the velocity correlation:

$$\langle u^P_\alpha(0,t_1)u^P_\beta(\mathbf{r},t_2)\rangle=\frac{C_{Kol}\overline{\epsilon}^{2/3}}{\pi}\int_0^\infty \frac{kdk}{(k^2+k_0^2)^{4/3}}\exp(-\gamma_k|t_1-t_2|)\int_0^{2\pi}d\phi\left[\frac{r_\alpha r_\beta}{r^2}\cos^2\phi+\left(\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)\sin^2\phi\right]\exp(ikr\cos\phi)$$
$$\times\exp\left\{-C_{Kol}\overline{\epsilon}^{2/3}\tau_S^2k^2\int_0^\infty\frac{\{F(s,t_1,t_2)+G(s,t_1,t_2)[J_0(sr)+J_2(sr)\cos 2\phi]\}sds}{(k_0^2+s^2)^{4/3}(1+\gamma_s\tau_S)}\right\},\tag{5.7}$$

where ϕ is the angle between $\mathbf{k}$ and $\mathbf{r}$,

$$F(s,t_1,t_2)=f(s,t_1)+f(s,t_2),$$
$$G(s,t_1,t_2)=f(s,t_1-t_2)-F(s,t_1,t_2)\tag{5.8}$$

and

$$f(s,t)=1-e^{-|t|/\tau_S}-\frac{e^{-\gamma_s|t|}-e^{-|t|/\tau_S}}{1-\gamma_s\tau_S}.\tag{5.9}$$

The effect of trajectory separation is contained in the last line of Eq. (5.7). We see that the contribution, which leads to finite divergence of the correlation $\langle u^P_\alpha(0,t_1)u^P_\beta(\mathbf{r},t_2)\rangle$, is the ϕ dependence of this factor. The remaining ϕ dependence, contained in the second line of this equation, is simply the factor $\mathbf{k}_\perp\mathbf{k}_\perp\exp(i\mathbf{k}\cdot\mathbf{r})$ arising in the Fourier transform of Eq. (3.1), and would give by itself zero divergence.

The argument of the exponential in the last line of Eq. (5.7), for fixed τ_S/τ_L, is $O(\rho^{-2})$, so that we may try a Taylor expansion. However, as it happened with Eqs. (4.21) and (4.31), the resulting integrals in k diverge. We therefore keep in the exponential the time independent piece of its argument, and expand the remnant, which, to leading order in ρ, gives the following expression:

$$1-C_{Kol}\overline{\epsilon}^{2/3}\tau_S^2k^2\cos 2\phi\exp\left\{-\int_0^\infty\frac{C_{Kol}\overline{\epsilon}^{2/3}\tau_S^2k^2sds}{(k_0^2+s^2)^{4/3}(1+\gamma_s\tau_S)}\right\}\int_0^\infty\frac{G(s,t_1,t_2)J_2(sr)sds}{(k_0^2+s^2)^{4/3}(1+\gamma_s\tau_S)}$$
$$=1-C_{Kol}\overline{\epsilon}^{2/3}\tau_S^2k^2\cos 2\phi\exp\left\{-\frac{3C_{Kol}\overline{\epsilon}^{2/3}\tau_S^2k^2}{2k_0^{2/3}}\right\}\int_0^\infty\frac{G(s,t_1,t_2)J_2(sr)sds}{(k_0^2+s^2)^{4/3}(1+\gamma_s\tau_S)}\tag{5.10}$$

plus terms that would lead to a divergence free contribution to $\langle u^P_\alpha(0,t_1)u^P_\beta(\mathbf{r},t_2)\rangle$ and would disappear from Eq. (5.2). Substituting into Eq. (5.7) and then back into Eqs. (5.4) and (5.3), we find, after carrying out the time integrals and the integral in ϕ,

$$D^{(1)}_{\alpha\beta}=\frac{8C_{Kol}^{3/2}\overline{\epsilon}\tau_S^2}{\rho}\int_0^\infty x^{-1/3}dx\exp\left\{-\frac{3S_l^2x^2}{2r^2}\right\}\int_0^\infty dyy^{-5/3}J_2(y)\left[1-\frac{1}{2[1+(x/y)^{2/3}]}\right]$$
$$\times\left[\delta_{\alpha\beta}\left(\frac{1}{2}J_0(x)-J_2(x)+\frac{1}{2}J_4(x)\right)-\frac{r_\alpha r_\beta}{r^2}J_4(x)\right],\tag{5.11}$$

and it is possible to see that $D^{(0)}_{\alpha\beta}$ is given by the same expression valid for a fluid parcel, i.e., by Eq. (3.17),

$$D^{(0)}_{\alpha\beta}(\mathbf{r})=\frac{4\alpha_{7/3}C_{Kol}^{1/2}\overline{\epsilon}^{1/3}}{\rho}r^{4/3}\left[\frac{r_\alpha r_\beta}{r^2}+\frac{7}{3}\left(\delta_{\alpha\beta}-\frac{r_\alpha r_\beta}{r^2}\right)\right].\tag{5.12}$$

The physical content of the expansion leading to Eqs. (5.11) and (5.12) can be clarified, noticing that, in a way perfectly analogous to Eqs. (4.10) and (4.11), the generating functional $Z[\mathbf{J}]$ entering Eq. (5.5) can be written as

$$\left\langle\exp\left(i\mathbf{k}\cdot\int dt[\mathbf{u}^L(0,t)\overline{J}_{t_1}(t)-\mathbf{u}^L(\mathbf{r},t)\overline{J}_{t_2}(t)]\right)\right\rangle\sim\exp\left\{-\frac{\mathbf{kk}}{2}:\int dtdt'\ [\mathbf{U}(0,t-t')[\overline{J}_{t_2}(t)\overline{J}_{t_2}(t')+\overline{J}_{t_1}(t)\overline{J}_{t_1}(t')]\right.$$
$$\left.-2\mathbf{U}(\mathbf{r},t-t')\overline{J}_{t_1}(t)\overline{J}_{t_2}(t')]\right\}.$$

The argument in the exponential is in the form $\sim k^2(S_1^2(t_1-t_2)+S_2^2(t_1-t_2,r))$ with $S_{1,2}$ the one- and two-particle contributions to trajectory separation, with $S_1(0)=0$, $S_1(\infty)\sim S_l$, and $S_2\sim C_{Kol}^{1/2}(\overline{\epsilon r})^{1/3}\tau_S$. Substituting into the definition of $D_{\alpha\beta}$ gives then, using $r\ll S_l$,

$$D\sim\int dt\int kdkU_k e^{-k^2S_1^2(t)-\gamma_k t}[1-\exp(-k^2S_2^2(t,r)-i\mathbf{k}\cdot\mathbf{r})]\sim\int kdkU_k\gamma_k^{-1}(1-e^{-i\mathbf{k}\cdot\mathbf{r}})+\int k^3dkU_k\gamma_k^{-1}C_{Kol}(\overline{\epsilon r})^{2/3}\tau_S^2e^{-k^2S_l^2},$$

which, using Eq. (4.9), is $\sim(C_{Kol}^{1/2}\overline{\epsilon}^{1/3}/\rho)[r^{4/3}+\rho^2S_i^{4/3}(r/S_l)^{2/3}]$. Thus, the origin of the $D^{(1)}$ in the two-particle contribution to trajectory separation is confirmed, together with its being generated at the k_0-dependent scale S_l. This is of course confirmed by direct analysis of Eq. (5.11); the integral is dominated by $k=x/r\sim S_l^{-1}$ and we obtain, for $r>S_i$,

$$\frac{D^{(1)}}{D^{(0)}}\sim\frac{\rho^2S^{4/3}}{(rS_l)^{2/3}}<\rho^{2/3}\left(\frac{\tau_S}{\tau_L}\right)^{1/3}. \tag{5.13}$$

Thus, for $\tau_S<\rho^{-2}\tau_L$, $D^{(1)}<D^{(0)}$ and Eq. (5.3) applies.

Using the relation $\int_0^\infty dxx^{-1/3}e^{-\alpha x^2}J_4(x)=[\Gamma(7/3)/2^5\alpha^{7/3}\Gamma(5)]M(7/3,5,-1/(4\alpha))$, with $M(a,b,x)$ the confluent hypergeometric function [40], we obtain, in general, from Eq. (5.11),

$$D^{(1)}_{\alpha\beta}=4\rho\alpha_{7/3}C_{Kol}^{1/2}\overline{\epsilon}^{1/3}S_i^{4/3}\left(\frac{r_\alpha r_\beta}{r^2}\tilde{D}(r/S_l)+\delta_{\alpha\beta}\hat{D}(r/S_l)\right), \tag{5.14}$$

where we can write

$$\tilde{D}(r/S_l)=-\frac{c\beta\ \ 2^{2/3}\Gamma(7/3)}{\alpha_{7/3}\Gamma(5)}\left(\frac{r^2}{6S_l^2}\right)^{7/3}M\left(\frac{7}{3},5,-\frac{r^2}{6S_l^2}\right),$$

$$\beta=\int_0^\infty dyy^{-5/3}J_2(y), \tag{5.15}$$

with $c\simeq1$ for $r\gg S_l$ and $c\simeq\frac{1}{2}$ for $r\ll S_l$; $\hat{D}$ will be shown not to contribute to the concentration correlations.

We can now calculate the probability $P^{(1)}(r)$. Substituting Eqs. (5.13) to (5.15) into Eq. (5.2), after a few manipulations, leads to

$$\partial_{\bar{r}}\bar{r}^{7/3}\partial_{\bar{r}}P^{(1)}=-\rho^2\bar{r}P^{(0)}\left(\frac{S_i}{S_l}\right)^{4/3}\left(\partial_{\bar{r}}^2(\tilde{D}(\bar{\mathbf{r}})+\hat{D}(\bar{r}))+\frac{1}{\bar{r}}\partial_{\bar{r}}(2\tilde{D}(\bar{r})+\hat{D}(\bar{r}))\right), \tag{5.16}$$

where $\bar{r}=r/S_l$. Hence, for $S_i\ll r\ll S_l$,

$$P^{(1)}(r)=-\rho^2P^{(0)}\left(\frac{S_i}{S_l}\right)^{4/3}\int_{r/S_l}^\infty dyy^{-7/3}[\tilde{D}(\infty)-y\partial_y(\tilde{D}(y)+\hat{D}(y))-\tilde{D}(y)]\simeq-\frac{3}{4}\rho^2P^{(0)}\tilde{D}(\infty)\left(\frac{S_i}{r}\right)^{4/3}. \tag{5.17}$$

Using Eq. (5.15) and the limiting form for the confluent hypergeometric function $M(a,b,-z)=[\Gamma(b)/\Gamma(b-a)]z^{-a}[1+O(z^{-1})]$ [40], we get the final result,

$$\Theta(r)=\overline{\theta^2}\left[1+\bar{\beta}\rho^2\left(\frac{S_i}{r}\right)^{4/3}\right], \tag{5.18}$$

where

$$\bar{\beta}=\frac{3\beta\Gamma(7/3)}{2^{2/3}\alpha_{7/3}\Gamma(8/3)}\simeq2.14. \tag{5.19}$$

In conclusion, we have a range of separations $S_i\ll r\ll S_l$, in which the fluctuation correlation grows with a power $-4/3$, to reach amplitude $\sim\rho^2$ at $r\sim S_i$.

The picture that arises is one of concentration fluctuations produced at scale S_l, by compressibility of the solid particle flow, and then transported to small scales and amplified by the incompressible part of the flow. The process is different from that of a passive scalar forced at large scale, due to the derivatives in the source term [and, in fact, the scaling exponent is different; compare with Eq. (3.36)]. This source term is basically $\nabla^2D^{(1)}(r)$, with $D^{(1)}(r)$ saturating at a constant for $r\gg S_l$ and going to zero in the opposite limit. From here, the $r^{-4/3}$ scaling of Eq. (5.18) arises by dimensional analysis.

What happens when $r\ll S_i$? At such short distances, the separation process is ballistic and we cannot use a diffusive approximation anymore. In Ref. [24], it is suggested that the correlation buildup should stop only because of discreteness effects or because of the Brownian motion of the solid particle. Actually, extrapolating the results of the present paper to the real turbulence regime $\rho=O(1)$, there is good reason to think that, for $\tau_S>\tau_\eta$, this buildup could stop much earlier, and precisely at $r\sim S_i$, which, for $\rho=O(1)$, coincides with the size of vortices with eddy turnover time τ_S.

At separations below S_i, Eq. (5.18) ceases to be valid, and full analysis of the distribution $P(\mathbf{r},\Delta_r\mathbf{v})$ is needed. A singularity of $P(r)$ at $r=0$ would require focusing of $\Delta_r\mathbf{v}$ along $\mathbf{r}$ for $r\ll S_i$; the mechanism is sketched in Fig. 1. This means that $P(\mathbf{r},\Delta_r\mathbf{v})$ itself should develop, as $r\to0$, a singularity at θ, where θ is the angle between $\Delta_r\mathbf{v}$ and $\mathbf{r}$. The necessary trajectory focusing can be produced only by the compressible part of $\mathbf{v}$. However, for $r<S_i$, the production term for the compressible part of $\mathbf{v}$ can be estimated directly from the second term in Eq. (5.10), to be $O(\rho^{-2})$ relative to the rest, and is able to act only for a time τ_S in the ballistic region. Hence, $P(\mathbf{r},\overline{\mathbf{v}})$ must be singular before this region is reached. However, for $r>S_i$, where the diffusive approximation works, the asymmetry of $P(\mathbf{r},\overline{\mathbf{v}})$, associated with compressibility of the flow, can be estimated from $\langle\overline{v_\alpha v_\beta}\rangle^{(1)}/\langle\overline{v_\alpha v_\beta}\rangle^{(0)}\sim D^{(1)}/D^{(0)}=O(\rho^{-2/3})$ so that singularities are not expected in $P(\mathbf{r},\overline{\mathbf{v}})$, for $\theta=0$ and $r\geqslant S_i$ either. The conclusion is that a plateau for $\Theta(r)$ should be present at $r<S_i$.

VI. SOLID TRACERS: ERGODIC PROPERTIES

One of the consequences of the compressibility of the velocity field $\mathbf{v}(\mathbf{x},t)$ is that the ergodic property is not satisfied anymore: velocity moments calculated along solid particle trajectories differ from those obtained from spatial averages. As mentioned before, physical intuition suggests that solid particles should privilege in their motion certain regions of the fluid with respect to the others (namely, hyperbolic with respect to elliptic regions). It is difficult, however, to translate this into a statement on the form of the PDF for the velocity $\mathbf{u}^P$.

We have at our disposal the equations satisfied by the velocity field $\mathbf{u}^P$. It is possible therefore to calculate its moments and to reconstruct its PDF. We consider the case of zero gravity $\mathbf{u}_G=0$ and τ_S/τ_L small. As in the analysis of the concentration fluctuations, all nonuniversal effects associated with the large scales of the flow are thus eliminated from the problem. From definition of $\mathbf{u}^P$ and Eqs. (4.1) and (4.2), we obtain the following set of equations, valid to lowest order in ρ^{-1}:

$$[\partial_t+\tilde{\mathbf{u}}(\mathbf{x},t)\cdot\boldsymbol{\nabla}]\mathbf{u}^P(\mathbf{x},t)+\int dy^2\gamma(\mathbf{x}-y)\mathbf{u}^P(\mathbf{y},t)=\int d^2y\,h(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}(\mathbf{y},t),$$

$$\tilde{\mathbf{u}}(\mathbf{x},t)\equiv\mathbf{u}^P(\mathbf{x},t)-\mathbf{v}^P(\mathbf{x},t)=\int_{-\infty}^t d\tau\exp\left(-\frac{t-\tau}{\tau_S}\right)\dot{\mathbf{u}}^P(\mathbf{x},t),\tag{6.1}$$

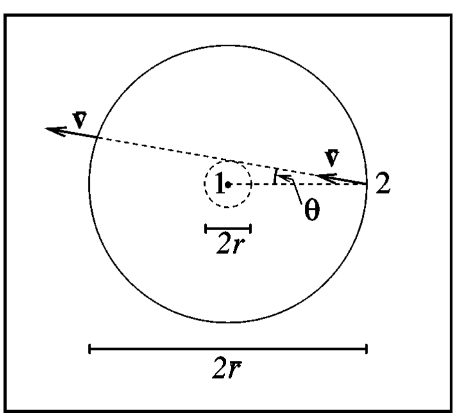

FIG. 1. Sketch of the behavior of particle relative velocities inside a domain of size $R<S_i$. Particle 2 moves with respect to particle 1 at constant velocity. In order for the particle density to diverge as $r\to0$, it is necessary that the distribution of velocities be peaked too at $\theta=0$.

which differs from the analogous equation for $\mathbf{u}^L$ because of the non-volume-preserving advection term $\tilde{\mathbf{u}}\cdot\nabla\mathbf{u}^P$. From here we can carry on standard field theoretical perturbation theory, either by the Martin-Siggia-Rose formalism [27], or working directly with Eq. (6.1). The building blocks of the diagrammatic expansion are shown in Fig. 2, and are the propagator $G_{\mathbf{k}\alpha\beta}$,

$$G_{\mathbf{k}\alpha\beta}u^P_{\mathbf{k}\beta}(t)=\int_{-\infty}^t d\tau\exp[-\gamma_k(t-\tau)]\frac{k_\alpha k_\beta}{k^2}u^P_{\mathbf{k}\beta}(\tau),\tag{6.2}$$

the correlator $U^P_{\mathbf{k}\alpha\beta}(t)$,

$$U^P_{\mathbf{k}\alpha\beta}(t)=\frac{k^\perp_\alpha k^\perp_\beta}{k^2}\frac{4\pi C_{Kol}\overline{\epsilon}^{2/3}}{(k^2+k_0^2)^{4/3}}\exp(-\gamma_k|t|)\tag{6.3}$$

and the vertex $\Gamma_{\mathbf{k}\alpha\beta\gamma}$,

$$\Gamma_{\mathbf{k}\alpha\beta\gamma}u^P_{\mathbf{p}\beta}u^P_{\mathbf{s}\gamma}(t)=i\lambda s_\beta\delta_{\alpha\gamma}\delta(\mathbf{k}+\mathbf{p}+\mathbf{s})\int_{-\infty}^t d\tau\times\exp\left(-\frac{t-\tau}{\tau_S}\right)u^P_{\mathbf{s}\gamma}(\tau)\partial_\tau u^P_{\mathbf{p}\beta}(\tau),\tag{6.4}$$

where the coefficient $\lambda=1$ is introduced, as in Eq. (2.15), only for the purpose of book keeping. To lowest order in λ, the correlations for the fields $\mathbf{u}^P(\mathbf{x},t)$ and $\mathbf{u}^L(\mathbf{x},t)$ are trivially equal. To higher orders, differences arise, which would not lead, if $\boldsymbol{\nabla}\cdot\tilde{\mathbf{u}}=0$, to differences between the one-point PDFs for $\mathbf{u}^P$ and $\mathbf{u}^L$ (see also Ref. [41]). In our case, this is not so, and the difference between the moments of the two PDFs can be calculated in perturbation theory; to $O(\lambda^n)$,

$$\langle(u^P)^m\rangle^{(n)}=\int\frac{d^2k_1}{(2\pi)^2}\cdots\frac{d^2k_m}{(2\pi)^2}\langle u_{\mathbf{k}_1}(t)\cdots u_{\mathbf{k}_m}(t)\rangle^{(n)},\tag{6.5}$$

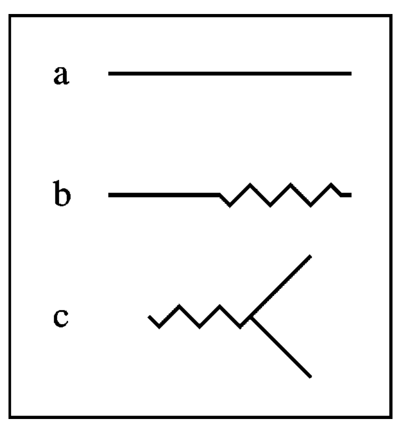

FIG. 2. Feynman diagrams for the propagator $G_{\mathbf{k}\alpha\beta}$ (a), for the correlator $U^P_{\mathbf{k}\alpha\beta}$ (b), and the vertex $\Gamma_{\mathbf{k}\alpha\beta\gamma}$ (c).

where $\langle u_{\mathbf{k}_1}(t)\cdots u_{\mathbf{k}_m}(t)\rangle^{(n)}$ is the sum of the Feynman diagrams with m outgoing velocity lines and n vertices. Because of symmetry under space reflection, the lowest-order contributions are $O(\lambda^2)$; the corresponding diagrams are shown in Fig. 3 and lead to corrections to the velocity second and fourth moments. In order to check for the presence of divergences in loop diagrams, we carry on power counting on Eq. (6.1). Rescaling space and time as in Eq. (2.16), we find $[\lambda]=0$, implying the possibility of logarithmic divergences. Now, a perturbation expansion in λ of Eq. (6.1) ceases to be sensible at scales below the length S_i defined in Eq. (4.9). From Eq. (4.13), the effective decay rate for the field $\mathbf{u}^P$ appears to be

$$\gamma_k^P=\begin{cases}\gamma_k, & kS_i\ll 1\\ u_S k, & kS_i\gg 1,\end{cases} \tag{6.6}$$

and this expression should be substituted for γ in Eqs. (6.1)–(6.3). For $kS_i\gg 1$, γ_k^P is just the inverse of the crossing time of an eddy of size k^{-1}. In this large k range, the appropriate scaling for the frequency should be, instead of the one provided by Eq. (2.17), which leads to $[\lambda]=0$, the following one:

$$[t]=1,\quad [\lambda]=-[u]=-\tfrac{1}{2}. \tag{6.7}$$

The change of scaling in γ^P is therefore sufficient to regularize the divergent diagrams, providing an effective ultraviolet cutoff at $k=S_i^{-1}$.

We calculate explicitly the loop diagram in Fig. 3, and the corresponding correction to $\langle(u^P)^2\rangle$,

$$\begin{aligned}\langle(u^P)^2\rangle^{(2)}=&\int\frac{d^2p}{(2\pi)^2}\frac{d^2s}{(2\pi)^2}\int_{-\infty}^0 dt_1\int_{-\infty}^0 dt_2\int_{-\infty}^{t_1}d\tau_1\int_{-\infty}^{t_2}d\tau_2\exp\left(-\gamma_k^P(t_1+t_2)-\frac{t_1+t_2-\tau_1-\tau_2}{\tau_S}\right)\\&\times\delta_{\alpha\beta}\partial_{\tau_1}\partial_{\tau_2}s_\gamma s_\delta[U^P_{\mathbf{p}\gamma\delta}(\tau_1-\tau_2)U^P_{\mathbf{s}\alpha\beta}(t_1-t_2)+U^P_{\mathbf{p}\gamma\beta}(\tau_1-t_2)U^P_{\mathbf{s}\alpha\delta}(t_1-\tau_2)\\&+U^P_{\mathbf{p}\alpha\gamma}(t_1-\tau_2)U^P_{\mathbf{s}\alpha\beta}(\tau_1-t_2)+U^P_{\mathbf{p}\alpha\beta}(t_1-t_2)U^P_{\mathbf{s}\gamma\delta}(\tau_1-\tau_2)],\end{aligned}$$

where $\mathbf{k}=-\mathbf{p}-\mathbf{s}$. The time integrations can be carried out at once and, after some algebra, we reach the following result:

$$\begin{aligned}\langle(u^P)^2\rangle^{(2)}=&2\int\frac{d^2p}{(2\pi)^2}\frac{d^2s}{(2\pi)^2}\frac{U_p^PU_s^P\gamma_p^P(\mathbf{p}_\perp\cdot\mathbf{s})^2}{(\gamma_k^P+\gamma_p^P+\gamma_s^P)(\gamma_k^P+\gamma_s^P+\tau_S^{-1})(\gamma_p^P+\tau_S^{-1})\gamma_k^Pp^2}\Bigg[-\gamma_p^P\tau_S(\gamma_k^P+\gamma_p^P+\gamma_s^P+\tau_S^{-1})\\&+\frac{(\mathbf{p}\cdot\mathbf{s})}{s^2}\frac{\gamma_s^P}{\gamma_s^P+\tau_S^{-1}}(\gamma_k^P+\gamma_s^P-\tau_S^{-1})\Bigg],\end{aligned} \tag{6.8}$$

where $U_k=U_{\mathbf{k}\alpha\alpha}(0)$. As predicted in the discussion leading to Eqs. (6.6) and (6.7), substituting $\gamma^P\to\gamma$ would lead to a logarithmically divergent integral. Comparing with Eq. (4.9), we see that this integral receives contribution from wave vectors in the range $[S^{-1},S_i^{-1}]$, i.e., from those eddies fast enough for the particles to be unable to respond to their velocity field, but still sufficiently slow for trajectory separation to be considered a perturbation. To find the leading behavior in S_i^{-1}, the integral can be rewritten, after the change of variables $y=(\gamma_{(ps)^{1/2}}\tau_S)^{-1}$, $z=p/q$, in the form

$$\begin{aligned}\langle(u^P)^2\rangle^{(2)}=&\frac{3u_S^2}{16\pi^3\rho^2}\int_0^{2\pi}d\phi\int_0^\infty\frac{dz}{z}\int_{\rho^{-2}}^\infty\frac{dy}{y}\frac{\sin^2\phi}{(\bar{p}^{2/3}+\bar{s}^{2/3}+\bar{k}^{2/3})(\bar{k}^{2/3}+\bar{s}^{2/3}+y)(\bar{p}^{2/3}+y)\bar{k}^{2/3}\bar{p}^{2/3}}\Bigg[-(\bar{p}^{2/3}+\bar{s}^{2/3}+\bar{k}^{2/3}+y)\\&+\frac{y\cos\phi}{\bar{s}^{2/3}+y}(\bar{s}^{2/3}+\bar{k}^{2/3}-y)\Bigg]+O(\rho^{-2}),\end{aligned} \tag{6.9}$$

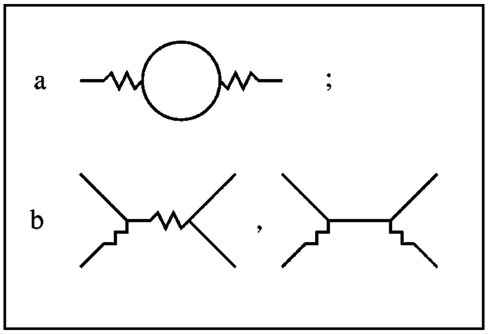

FIG. 3. Feynman diagrams providing the lowest order correction to $\langle (u^P)^2\rangle$ (a) and $\langle (u^P)^4\rangle$ (b).

where $\overline{\mathbf{k}}=(ps)^{-1/2}\mathbf{k}$, $\overline{\mathbf{p}}=(ps)^{-1/2}\mathbf{p}$, $\overline{\mathbf{s}}=(ps)^{-1/2}\mathbf{s}$, $\cos\phi=\overline{\mathbf{p}}\cdot\overline{\mathbf{s}}$. We obtain then the final result,

$$\langle (u^P)^2\rangle^{(2)}=\frac{\overline{\eta}\ln\rho}{\rho^2}u_S^2, \tag{6.10}$$

where

$$\begin{aligned}\overline{\eta}=&\frac{3}{8\pi^3}\int_0^{2\pi}d\phi\int_0^\infty\frac{dz}{z}\,\frac{z^{1/3}\sin^2\phi}{(z+z^{-1}+2\cos\phi)^{1/3}}\\&\times\Bigg[-\frac{1}{(z+z^{-1}+2\cos\phi)^{1/3}+z^{1/3}}\\&+\frac{\cos\phi}{z^{1/3}+z^{-1/3}+(z+z^{-1}+2\cos\phi)^{1/3}}\Bigg]\\\simeq&-0.32\end{aligned} \tag{6.11}$$

is evaluated by numerical integration. The correction to the velocity amplitude is negative. In the presence of inertia, solid tracers prefer therefore to lie in regions of the flow where the turbulent velocity is smaller.

Extrapolating Eq. (6.10) to $\rho=O(1)$ suggests that $\langle (u^P)^2\rangle-u_T^2\sim u_S^2$. We can have some idea of what we should expect for dominant gravity $u_S<u_G$, from dimensional analysis of Eq. (6.8). In this case $\gamma_k\to u_Gk$, the inverse sweep time due to the particle fall, and we would find $\langle (u^P)^2\rangle^{(2)}\sim k^4U_k^2/u_G^2$, with $k^{-1}\sim u_G\tau_S$ giving the transition to the small scales for which the sweep time is shorter than τ_S, and to which the particles are unable to respond. From here we find $\langle (u^P)^2\rangle-u_T^2\sim(u_S/u_G)^{2/3}u_S^2$ and we see that gravity reduces the amount of nonergodicity of the solid particle flow.

VII. CONCLUSIONS

Consideration of a finite correlation time in the transport by a random velocity field has allowed analysis of a series of issues. We summarize the main results in the following.

(i) The self-diffusion of a fluid parcel obeys linear scaling in the inertial range (as it should) with a universal constant $C_0=C_{Kol}^{3/2}[\hat{\rho}\rho/(\hat{\rho}-\rho)]\ln\hat{\rho}/\rho$ [see Eqs. (3.7)–(3.10)], which is sensitive both to the ratio of the eddy turnover and lifetime, and to the rate of eddy velocity decorrelation at times much shorter than the eddy lifetime. A quadratic maximum (at least), at time separation equal to zero, is necessary for C_0 to remain finite. (An exponential time correlation, for instance, would not satisfy this condition.) This sensitivity on the short time behavior of time correlation was not observed in any of the other transport processes considered in the present paper.

(ii) The relative diffusion of a pair of fluid parcels, exhibits (again as it should) Richardson and normal diffusion behavior, respectively, for coordinates and velocities. The PDF for relative separation is a stretched exponential with exponent $\frac{2}{3}$ [see Eq. (3.21)] and it is possible to express the universal constants c and $\tilde{c}$ entering, respectively, coordinate and velocity dispersion, in terms of the parameter ρ. Precisely, $c\simeq 0.748C_{Kol}^{3/2}/\rho^3$ and $\tilde{c}\simeq 3.037C_{Kol}^{3/2}/\rho$ [see Eqs. (3.23) and (3.26)]. For the Batchelor constant, we obtain instead [see Eq. (3.39)] $B\simeq 11.32\rho$.

(iii) The correlation time τ_P for the fluid velocity sampled by a solid particle has a behavior consistent with previous analysis neglecting the structure of the turbulent inertial range [17,38]. Values of τ_P/τ_L, above unity are found for dominant inertia and $\tau_S\lesssim\tau_L$, with $\tau_P/\tau_L-1=O((\tau_S/\rho\tau_L)^{4/3})$ [see Eq. (4.25)]. On the contrary, in the case of dominant gravity, $\tau_P/\tau_L<1$ irrespective of the value of the ratio u_T/u_G between the turbulent and the fall velocity; specifically [see Eq. (4.28)], we find $\tau_P/\tau_L-1=O((u_Gk_0\tau_L)^2)$ for $u_G\ll u_L$, and $\tau_P/\tau_L=O((u_Gk_0\tau_L)^{-1})$ in the opposite case. For short times, the expected sublinear behavior for the fluid velocity along a solid particle trajectory is found: $\langle |u^P(x,t)-u^P(x,0)|^2\rangle\sim(\overline{\epsilon}u_At)^{2/3}$, with $A=G,S$ depending on whether gravity or inertia dominates [see Eqs. (4.16) and (4.19)].

(iv) The Eulerian correlation time τ_E (and by continuity, therefore, also τ_P, in the regime $\tau_S\gg\tau_L$) is shorter than its Lagrangian counterpart, with $\tau_E/\tau_L=1-2\rho^{-2}\ln\rho$ [see Eq. (4.34)]. Sweep produces a power-law decay of correlations between velocity increments in the form $S_{rr}(r,t)=\langle[u_r(\mathbf{r},t)-u_r(0,t)][u_r(\mathbf{r},0)-u_r(0,0)]\rangle$. More precisely, for time separations longer than the sweep time $T_{r^{-1}}$: $S_{rr}(r,t)\sim S_{rr}(r,0)(T_{r^{-1}}/t)^{4/3}$ [see Eq. (4.37)].

(v) In the absence of gravity, and for $\rho^2\tau_\eta\ll\tau_S\ll\rho^{-2}\tau_L$, the spectrum of concentration correlation induced by turbulence in a solid particle suspension, is universal and has power-law behavior for separations above the size S_i of an eddy that is crossed by a typical solid particle in a time equal to its lifetime. More precisely, $\overline{\theta}^{-2}\langle\theta(r)\theta(0)\rangle-1\simeq\rho^2(S_i/r)^{4/3}$ [see Eq. (5.18)].

(vi) The solid particle flow is nonergodic, with a difference between the fluid velocity sampled along a solid trajectory and the corresponding Eulerian average: $\langle (u^P)^2\rangle-\langle u^2\rangle=-(0.32\ln\rho/\rho^2)u_S^2$ [see Eqs. (6.10) and (6.11)]. Dimensional reasoning for $\rho=O(1)$ suggests that gravity should reduce this effect from $\langle (u^P)^2\rangle-u_T^2\sim u_S^2$ to $\langle (u^P)^2\rangle-u_T^2\sim(u_S/u_G)^{2/3}u_S^2$.

Analysis of some of these problems actually did not exploit the finite correlation time of the velocity field produced through Eq. (2.7). In particular, the process of fluid parcel relative dispersion was considered to the same order in ρ as in the Kraichnan model and, to this level, no information on the Lagrangian statistics was necessary. Finiteness of the correlation time had the only purpose to allow a meaningful definition of quantities such as C_{Kol} and $\overline{\epsilon}$.

In the case of the self-diffusion properties of fluid and solid particles, a finite correlation time and inclusion of the Lagrangian nature of time correlation was necessary from the start. Nonetheless, the only point in which analysis of the random velocity field could not be avoided, was to determine the dimensionless constant C_0 [14,29]; the diffusion exponents in the various cases were already available by dimensional reasoning.

Evaluation of the correlation time τ_P and analysis of concentration fluctuations and nonergodicity of particle trajectories [points (iv)–(vi)], instead, rested heavily on the fact that the correlation time was finite and on knowledge of the actual form of the random velocity field time correlation. The analysis confirmed the role of eddies with lifetime τ_S, already pointed out in Ref. [22].

Some comments are due on these last issues. As regards correlation times, they depend, in general, on nonuniversal aspects of the velocity statistics, and, in the present case, on the assumption that also the large scale statistics is defined along Lagrangian trajectories. In consequence of this, the Eulerian time of the flow resulted shorter than the Lagrangian correlation time. (Following Ref. [37], the Eulerian correlation feels, at the same time, the decorrelation from relative motion of the fluid, and the effect of eddy decay.) For $\tau_S \ll \tau_L$, the standard picture of inertia and gravity leading, respectively, to increase and decrease of the correlation time, however, was confirmed.

As regards concentration fluctuations, previous treatments of this problem, either were limited to the case of particles with Stokes time shorter than the Kolmogorov time of the flow [24], or neglected turbulent small scale structures altogether [23]. This was due to the difficulty in analyzing trajectory crossing effects on inertial range scales, associated with the need for a proper treatment of the Lagrangian time statistics. The fully kinetic treatment adopted here, in which the relative motion of individual solid particles is fully taken into account, in contrast to the fluid equation approach used in Ref. [24], together with the large ρ limit, is what allows treatment of the problem.

It should be mentioned that solid particle concentration fluctuations may be important in the process of rain formation. It is known that the settling rate of a suspension is enhanced in the presence of clumping of the heavy particles [42], and turbulence induced concentration fluctuations appear to be one of the important actors in the process [43]. Inclusion of the effect of gravity, on the same lines of the analysis carried on in Sec. IV would therefore be necessary.

As regards nonergodicity of the solid particle flow, it should be mentioned that this is a problem one has to deal with, before trying to extend standard Lagrangian transport models (in particular, the well mixedness hypothesis on which they are based [14]) to the case of solid particles.

An important aspect that must be stressed, in the calculation of both τ_P and the concentration correlation spectrum, is the role played by the localization length S_l. This length ceases to have a physical meaning for finite ρ, nonetheless, it fixes, in perturbation theory, the scale at which both fluctuations and the difference $\tau_P - \tau_L$ are generated. Notice that, in the case of concentration fluctuations, this occurs in spite of the fact that the concentration correlations are peaked at the inertial scale S_i.

Another peculiarity of the large ρ expansion is the multiplicity of space scales associated with eddies having time or velocity scales related to τ_S and $\mathbf{u}_S$ [see Eqs. (4.6) and (4.9)]. All of them collapse, for $\rho = O(1)$, on the size of a vortex with turnover time equal to τ_S. In real high Reynolds number turbulence, this is the saturation length expected for concentration fluctuation buildup, when τ_S is an inertial range quantity.

The parameters ρ and $\hat{\rho}$ are central to the extension of the Kraichnan model to finite correlation times. The situation of reference in real flows is the inverse cascade range of two-dimensional turbulence. An estimate of these parameters could be obtained using the leading ρ expressions provided by Eqs. (3.10), (3.23), (3.26), and (3.39), with the values of the constants C_0, c, $\tilde{c}$, and B obtained from DNS. For instance, assuming $\hat{\rho} = \infty$, comparison with the results presented in Ref. [32] would give $\rho \simeq 2$.

The results of the present paper have been obtained to leading order in ρ. To this order, no perturbative effects in the structure of random velocity fields are present, and the correlations for the Lagrangian velocity u^L obey Eq. (2.6). The parameters entering these correlations must, nonetheless, be considered as renormalized quantities in a renormalized statistical field theory. No claim on the nature of these renormalizations is made, apart from that, to lowest order, marginality of interactions suggests that correction to scaling be only logarithmic.

To this order in ρ, extension of the results to three dimensions presents no conceptual difficulties. In particular, the mechanism of production for concentration fluctuations, and for correlation time and PDF corrections, is not expected to suffer modifications. Whether a random velocity field model like the present one could be appropriate to describe transport by a three dimensional turbulence, laden with coherent structures and intermittency, is a different matter.

The present extension to finite correlation times of the Kraichnan model is perturbative in nature. Imposition of time statistics along Lagrangian trajectories had as consequence a non-Gaussian velocity field. This resulted in a field theoretical perturbation theory, with expansion parameter ρ^{-1}, which is somewhat different from other field theories arising from closure analysis of the Navier-Stokes equation. It would be interesting to understand the relation with such theories, in particular, with the quasi-Lagrangian approach described in Ref. [44] and following papers based on this work (see Ref. [45] and references therein).

There are situations in which the higher orders in ρ^{-1} become necessary. A relevant example could be the derivation of a turbulent closure: in this case, extension of the theory to realistic values of ρ could not be avoided. Related to this issue, is the calculation of the anomalous scaling exponents for a passive scalar advected by a random velocity field with finite correlation time. The analysis of pair diffusion carried on in Sec. III proceeded, at the end, as if the velocity field had zero correlation time. To lowest order in ρ^{-1}, the same zero-mode structure of the Kraichnan model is therefore expected [8]. To proceed in a consistent way, one should go to higher order, at the same time, in the passive tracer part of the problem and in the field theory for the velocity field. Such issues, concerning the nature of the field theoretical perturbation expansion, will be analyzed in a separate publication.

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